Some Variants of the Balancing Sequence

Thesis submitted to

National Institute of Technology Rourkela

in partial fulfilment of the requirements

of the degree of

Doctor of philosophy

in

Mathematics

by

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September 2016

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This is to certify that the work presented in this dissertation entitled *Some Variants of the Balancing Sequence* by *Akshaya Kumar Panda*, Roll No 512MA305, is a record of original research carried out by him under my supervision and guidance in partial fulfilment of the requirements of the degree of *Doctor of Philosophy* in *Mathematics*. Neither this dissertation nor any part of it has been submitted for any degree or diploma to any institute or university in India or abroad.

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Dedicated

To

My Parents

Declaration of Originality

I, *Akshaya Kumar Panda*, Roll Number 512MA305 hereby declare that this dissertation entitled *Some Variants of the Balancing Sequence* represents my original work carried out as a doctoral/postgraduate/undergraduate student of NIT Rourkela and, to the best of my knowledge, it contains no material previously published or written by another person, nor any material presented for the award of any other degree or diploma of NIT Rourkela or any other institution. Any contribution made to this research by others, with whom I have worked at NIT Rourkela or elsewhere, is explicitly acknowledged in the dissertation. Works of other authors cited in this dissertation have been duly acknowledged under the section "Reference". I have also submitted my original research records to the scrutiny committee for evaluation of my dissertation.

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Acknowledgement

First of all, I would like to express my sincere gratitude and gratefulness to my supervisor Prof. Gopal Krishna Panda for his effective guidance and constant inspiration throughout my research work. Above all he provided me unflinching encouragement and support in various ways which enriched my sphere of research knowledge.

I take the opportunity to express my thankfulness and gratitude to the members of my doctoral scrutiny committee Prof. B. Majhi, Prof. S. Chakravarty, Prof. J. Mohapatra and, Prof. K. C. Pati , who is also Head of the Department, for being helpful and generous during the entire course of this work.

I thank to the Director, National Institute of Technology, Rourkela for permitting me to avail the necessary facilities of the institution for the completion of this work.

I am thankful to the Ph.D. students Ravi, Sai, Sushree and Manasi for their help and support during my stay in the department and making it a memorable experience in my life.

I would like to acknowledge deepest sense of gratitude to my mother and elder brothers for their constant unconditional support and valuable suggestion and motivation to face all the problems throughout my academic career.

I owe a lot to my wife Sujata and son Anshuman for their patience in spite of the inevitable negligence towards them during the preparation of this thesis.

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Abstract

Balancing and cobalancing numbers admit generalizations in multiple directions. Sequence balancing numbers, gap balancing numbers, balancing-like numbers etc. are examples of such generalizations. The definition of cobalancing and balancing numbers involves balancing sums of natural numbers up to certain number and beyond the next or next to next number up to a feasible limit. If these sums are not exactly equal but differ by just unity then the numbers in the positions of balancing and cobalancing numbers are termed as almost balancing and almost cobalancing numbers. Almost balancing as well as almost cobalancing numbers are governed by pairs of generalized Pell's equation which are suitable alteration of the Pell's equations for balancing and cobalancing numbers respectively. Similar alterations in the system of Pell's equations of the balancing-like sequences result in a family of generalized Pell's equation pair and their solutions result in almost balancing-like sequences. Another generalization of the notion of balancing numbers is possible by evenly arranging numbers on a circle (instead of arranging on a line) and deleting two numbers corresponding to a chord so as to balance the sums of numbers on the two resulting arcs. This consideration leads to the definition of circular balancing numbers. An interesting thing about studying several variations in the balancing sequence is that such variations increase the possibility of their application in other areas of mathematical sciences. For example, some of the balancing-like sequences along with their associated Lucas-balancing-like sequences are very closely associated with a statistical Diophantine problem. If the standard deviation σ of N consecutive natural numbers is an integer then σ is twice some term of a balancing-like sequence and N, the corresponding term of the associated Lucas-balancing-like sequence. Also, these variations have many important unanswered aspects that would trigger future researchers to work in this area.

KEY WORDS: Diophantine equations, Fibonacci numbers, Balancing numbers, Co-balancing numbers, Balancing-like sequences, Pell's equation, standard deviation

Contents

Chapter 1

Introduction

Number theorists are like lotus-eaters – having once tasted of this food, they can never give it up.

Leopold Kronecker.

The theory of numbers has been a source of attraction to mathematicians since ancient time. The discovery of new number sequences and studying their properties is an all-time fascinating problem. Since numbers are often the first objects that non-mathematicians would think of when they think of mathematics, it may not be surprising that this area of mathematics has all time drawn more attention from a general audience than other areas of pure mathematics.

The most interesting and ancient number sequence is the Fibonacci sequence, commonly known as the numbers of the nature, discovered by the Italian mathematician Leonardo Pisano (1170-1250) who is known by his nick name Fibonacci. The sequence was developed to describe the growth pattern of a rabbit problem [45]. The problem is described as follows: "A pair of rabbits is put in a place surrounded by walls. How many pairs of rabbits can be produced from that pair in a year if it is assumed that every month each pair gives birth to a new pair which from the second month onwards becomes productive? " The answer to this problem can be explained with the help of the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 \cdots$. This sequence, in which each term is the sum of the two preceding terms, is known as Fibonacci sequence and has wide applications in the area of mathematics, engineering and science. The Fibonacci sequence also appears in biological settings such as branching in trees, arrangement of leaves on a stem, the fruitlets of pineapple, the flowering of artichoke, an uncurling fern and the arrangement of a pinecone [6, 12, 50].

Mathematically, the Fibonacci sequence can be defined recursively as $F_{n+1} = F_n + F_{n-1}$ with initial terms $F_0 = 0,\ F_1 = 1$. Another sequence with identical recurrence relation was discovered by the French mathematician Eduard Lucas $(1842 - 1891)$ in the year 1870 is known as Lucas sequence. More precisely, it is defined by the recurrence relation $L_{n+1} = L_n + L_{n-1}$ with the initial values $L_0 = 2,\ L_1 = 1$. The Lucas sequence shares many interesting relationship with Fibonacci sequence. The most general form of Lucas sequence is described by means of a linear binary recurrence given by $x_{n+1} = Ax_n + Bx_{n-1}$. Having obtained two independent solutions of this recurrence (one of which is not a constant multiple of other) corresponding to two different sets of initializations, any other sequence obtained from this recurrence can be expressed as a linear combination of the two given sequences [11, 23, 27].

The Lucas sequence corresponding to $A = 2$ and $B = 1$ results in the recurrence $x_{n+1} = 2x_n + x_{n-1}$ and one can get two independent sequences as $P_0 = 0,\ P_1 = 1$, $P_{n+1} = 2P_n + P_{n-1}$ and $Q_0 = 2, Q_1 = 1,\ \ Q_{n+1} = 2Q_n + Q_{n-1}$ for $n \geq 1$. The former sequence is well known as the Pell sequence while the latter one is called the associated Pell sequence. The importance of these two sequences lies in the fact that the ratios $Q_n/P_n\,, n = 1,2,\cdots$ are successive convergents in the continued fraction representation of $\sqrt{2}$. Their importance further lies in the fact that the products $P_nQ_n, n = 1,2,\cdots$ form another and interesting number sequence known as the sequence of balancing numbers [41].

In the year 1999, A. Behera and G. K. Panda [3] introduced the sequence of balancing numbers, of course being unaware of the relationship with Pell and associated Pell numbers. They call a natural number B, a balancing number if it satisfies the Diophantine equation $1 + 2 + \cdots + (B - 1) = (B + 1) + (B + 2) + \cdots + (B + R)$ for some natural number R, which they call the balancer corresponding to B. A consequence of the above definition is that, if B is a balancing number then $8B^2+1$ is a perfect square [3], hence B^2 is a square triangular number and the positive square root of $8B^2+1$ is called a Lucas-balancing number. An interesting observation about Lucas-balancing numbers is that, these numbers are associated with balancing numbers the way Lucas numbers are associated with Fibonacci numbers. The n^{th} balancing number is denoted by B_n and by convention $B_1 = 1$. The sequence of balancing numbers (also commonly known as the balancing sequence) satisfies the recurrence relation $B_{n+1} = 6B_n - B_{n-1}, n = 1,2,\cdots$ with the initial values $B_0 = 0, B_1 = 1$.

The balancing numbers coincide with numerical centers described in the paper "The house problem" by R. Finkelstein [14]. However, the detailed study of balancing numbers for the first time done by Behera and Panda [3] and further extensions were carried out in [26,33,38,43].

The n^{th} Lucas-balancing number is denoted by C_n, that is, $C_n = \sqrt{8B_n^2+1}$ and these numbers satisfy the recurrence relation $C_{n+1} = 6C_n - C_{n-1}$ which is identical with that of balancing numbers, however with different initial values $C_0 = 1, C_1 = 3$. There are many instances where Lucas-balancing numbers appears in connection with balancing numbers. The $(n+1)^{st}$ balancing number can be expressed as a linear combination of n^{th} balancing and n^{th} Lucas-balancing number, $B_{n+1} = 3B_n + C_n$ [3, 41]. Further Panda [34] proved that the $(m+n)^{th}$ balancing numbers can be written as $B_{m+n} = B_mC_n + C_mB_n$, which looks like the trigonometry identity $\sin(x+y) = \sin x\cos y + \cos x \sin y$. The balancing and Lucas-balancing numbers satisfy the identity $(C_m + \sqrt{8}B_m)^n = C_{mn} + \sqrt{8}B_{mn}$ which resembles the de-Moivre's theorem for complex numbers. The sequence of balancing numbers have sum formulas in which B_n behaves like an identity function. The sum of first n odd indexed balancing numbers is equal to B_n^2 and the sum of first n even indexed balancing numbers is equal to B_nB_{n+1}. In any of these sum formulas, if B_n is replaced by n, it reduces to the corresponding sum formula for natural numbers.

In the year 2012, Panda and Rout [37] introduced a class of sequences known as balancing-like sequences described by means of the binary recurrences $x_{n+1} = Ax_n - x_{n-1}, x_0 = 0, x_1 = 1$, $n = 1,2,\cdots$, where $A > 2$ is a positive integer. These sequences may be considered as generalizations of the sequence of natural numbers since the case $A = 2$ describes the sequence of natural numbers. Hence the balancing-like sequences are sometimes termed as natural sequences. The balancing sequence is a particular case of this class corresponding to $A = 6$. The balancing-like sequence corresponding to $A = 3$ coincides with the sequence of even indexed Fibonacci numbers.

If x is a balancing-like number, that is, a term of a balancing-like sequence corresponding to some given value of A then $Dx^2 + 1$, where $D = \frac{A^2-4}{4}$, is a perfect rational square and $y = \sqrt{Dx^2 + 1}$, is called a Lucas-balancing-like number. The sequence $\{y_n\}_{n=1}^{\infty}$, where $y_n = \sqrt{Dx_n^2 + 1}$ is an integer sequence if A is even and is connected with the balancing-like sequence $\{x_n : n = 1,2,\cdots\}$ the way Lucas-balancing sequence is connected with the balancing sequence. The identity $x_{m+n} = x_m y_n + y_m x_n$ [see 37] is the generalization of $B_{m+n} = B_m C_n + C_m B_n$. Further, the identity $(y_m + \sqrt{D}x_m)^n = y_{mn} + \sqrt{D}x_{mn}$ [37] is known as the de-Moivre's theorem for the balancing-like sequences. The identities $x_1 + x_3 + \cdots + x_{2n-1} = x_n^2$ and $x_2 + x_4 + \cdots + x_{2n} = x_n x_{n+1}$ [37] confirm the resemblance of balancing-like sequences with the sequence of natural numbers.

The Fibonacci sequence is enriched with an important property. If m and n are natural numbers and m divides n then F_m divides F_n. A sequence with this property is called a divisibility sequence. The converse is also true, that is, if F_m divides F_n then m divides n and hence the Fibonacci sequence is a strong divisibility sequence. The sequence of balancing numbers is also a strong divisibility sequence [39]. Panda [37] showed that all the balancing-like sequences are also strong divisibility sequences.

The balancing sequence is closely associated with another number sequence namely, the sequence of cobalancing numbers (also known as the cobalancing sequence). By definition, a cobalancing number b (with cobalancer r) is a solution of the Diophantine equation $1 + 2 + \cdots + b = (b + 1) + \cdots + (b + r)$ [32]. Thus, if b is a cobalancing number then $8b^2 + 8b + 1$ is a perfect square [32] or equivalently, the pronic number $b(b + 1)$ is triangular. The positive square root of $8b^2 + 8b + 1$ is called a Lucas-cobalancing number.

The n^{th} cobalancing number is denoted by b_n and the cobalancing sequence satisfies the non-homogeneous binary recurrence $b_{n+1} = 6b_n - b_{n-1} + 2$ with initial values $b_0 = b_1 = 0$. All the cobalancing numbers are even while the balancing numbers are alternatively odd and even. The n^{th} Lucas-cobalancing number is denoted by c_n and these numbers satisfy a recurrence relation identical with that of balancing numbers. More precisely, the Lucas-cobalancing numbers satisfy $c_{n+1} = 6c_n - c_{n-1}$, with initial values $c_1 = 1, c_2 = 7$. The Lucas-cobalancing numbers involve in the one step shift formula of cobalancing numbers, namely $b_{n+1} = 3b_n + c_n + 1, n = 1,2, \cdots$ [41].

There is an interesting observation about the balancing sequence. Behera and Panda [3] proved that any three consecutive terms of the balancing sequence are approximately in geometric progression. In particular, they proved that $B_n^2 = B_{n-1}B_{n+1} + 1$. Subsequently, Panda and Rout [37] showed that similar results are also true for all balancing-like sequences. So far as the cobalancing sequence is concerned, there is a slight disturbance in this pattern, the three terms in approximate geometric progression being $b_{n-1}, b_n - 1$ and b_{n+1}; in particular, $(b_n - 1)^2 = b_{n-1}b_{n+1} + 1$ [41, p.39].

There is a big association of balancing and cobalancing numbers with triangular numbers. The defining equations of balancing and cobalancing numbers involve triangular numbers only. Denoting he n^{th} triangular number by T_n ($T_n = n(n + 1)/2$), these equations can be written as $T_{B-1} = T_{B+R} - T_B$ and $T_b = T_{b+r} - T_b$ respectively. Further, if B is a balancing number with balancer R then B^2 is the triangular number T_{B+R}. For each natural number n, B_nB_{n+1} and $\frac{B_nB_{n+1}}{2}$ are triangular numbers and thus the triangular number B_nB_{n+1} is pronic, that is, a product of two consecutive natural numbers. Also, $B_{n-1}B_n = b_n(b_n + 1)$ so that $b_n(b_n + 1)$ is triangular as well as pronic and thus, $b_n = \lfloor\sqrt{B_{n-1}B_n}\rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor function. Lastly, for every n, the number b_nb_{n+1} is also triangular.

The sequences of balancing and cobalancing numbers are very closely associated with each other. Panda and Ray [41] proved that all cobalancing numbers are balancers and all cobalancers are balancing numbers. More precisely, the n^{th} cobalancing number is the n^{th} balancer while the n^{th} balancing number is the $(n + 1)^{st}$ cobalancer.

The balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers are related to Pell and associated Pell numbers in many ways [35]. The n^{th} balancing number is product of the n^{th} Pell number and n^{th} associated Pell number and is also half of the $2n^{th}$ Pell number. Every associated Pell number is either a Lucas-balancing or a Lucas-cobalancing number. More specifically, $Q_{2n} = C_n$ and $Q_{2n-1} = c_n,\ n = 1,2,\cdots$. Further, the sum of first $2n - 1$ Pell numbers is equal to the sum of the n^{th} balancing number and its balancer and the sum of first $2n$ Pell numbers is equal to the sum of $(n + 1)^{st}$ cobalancing number and its cobalancer. The sum of first n odd terms of Pell sequence is equal to the n^{th} balancing number, while the sum of its first n even terms is the $(n + 1)^{st}$ cobalancing number.

The Pell and associated Pell sequences are solutions of the Diophantine equations $y^2 - 2x^2 = \pm 1$, where the values of x correspond to Pell numbers while the values of y correspond to associated Pell numbers. A Diophantine equation of the form $y^2 - dx^2 = N$ where d is a non-square positive integer and $N \neq 0,1$ is called a generalized Pell's equation. The case $N = 1$ corresponds to a Pell's equation. Certain integer sequences are better described by means of Pell's equations. The balancing and the Lucas-balancing sequences are solutions of the Pell's equation $y^2 - 8x^2 = 1$, the former corresponds to the values of x while the latter corresponds to the values of y. It is well-known that if b is a cobalancing number then $8b^2 + 8b + 1$ is a perfect square, say equals y^2 and the substitution $x = 2b + 1$ reduces $8b^2 + 8b + 1 = y^2$ to $y^2 - 2x^2 = -1$. For even values of A, the balancing-like and the Lucas-balancing-like sequences are solutions of $y^2 - Dx^2 = 1$, where $A = 2K$ and $D = K^2 - 1$ (which is never a perfect square if $A > 2$), while for odd values of A, these numbers appears in the solutions of the generalized Pell's equation $y^2 - (A^2 - 4)x^2 = 4$.

While defining balancing numbers, a number is deleted and hence a gap is created in the list of first m (m is arbitrary and feasible) natural numbers so that the sum of numbers to the left of the deleted number is equal to the sum to its right. In case of cobalancing numbers, sums are balanced without deleting any number. In the year 2012, as generalizations of balancing and cobalancing numbers, Rout and Panda [38, 43] introduced a new class of number sequences known as the sequences of gap balancing numbers. Instead of deleting one number as in case of balancing numbers, they considered deleting k numbers from the first m (m is arbitrary and feasible) natural numbers so that the sum of numbers to the left of these deleted numbers is equal to the sum to their right. If k is odd they call the median of the deleted numbers, a k-gap balancing number; if k is even, then this median is fractional and they call twice the median, a k-gap balancing number.

The concept of balancing and cobalancing numbers has been generalized in many directions. In 2007, Panda [33] introduced sequence balancing and cobalancing numbers using any arbitrary sequence $\{a_m\}_{m=1}^{\infty}$ of real numbers instead of natural numbers. A member a_n of this sequence is called a sequence balancing number if $a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + \cdots + a_{n+r}$ for some natural number r. Similarly a_n is called a sequence cobalancing number if $a_1 + a_2 + \cdots + a_n = a_{n+1} + \cdots + a_{n+r}$ for some natural number r. Panda [33] proved that there is no sequence balancing number in the Fibonacci sequence and $F_1 = 1$ is the only sequence cobalancing number in this sequence.

Panda [33] called the sequence balancing and cobalancing numbers of the sequence $\{n^k\}_{n=1}^{\infty}$ as higher order balancing and cobalancing numbers respectively. The case $k = 1$

corresponds to balancing and cobalancing numbers respectively. He also called the higher order balancing and cobalancing numbers corresponding to $k = 2$ as balancing squares and cobalancing squares. For $k = 3$, he called these numbers as balancing cubes and cobalancing cubes. In [33] he proved that no balancing or cobalancing cube exists and further conjectured that, no higher order balancing or cobalancing number exists for $k > 1$. This conjecture has neither been proved nor disproved till today.

Behera et al. [4] further generalized the notion of higher order balancing numbers. They considered the problem of finding quadruples (n, r, k, l) in positive integers with $n \geq 2$ satisfying the equation $F_1^k + F_2^k + \cdots + F_{n-1}^k = F_{n+1}^l + F_{n+2}^l + \cdots + F_{n+r}^l$ and conjectured that the only quadruple satisfying the above equation is $(4, 3, 8, 2)$. In this connection Irmak [18] studied the equation $B_1^k + B_2^k + \cdots + B_{n-1}^k = B_{n+1}^l + B_{n+2}^l + \cdots + B_{n+r}^l$ in powers of balancing numbers and proved that no such quadruple (n, r, k, l) in positive integers with $n \geq 2$ exists.

Komatsu and Szalay [21] studied the existence of sequence of balancing numbers using binomial coefficients. They considered the problem of finding x and $y \geq x + 2$ satisfying the Diophantine equation $\binom{0}{k} + \binom{1}{k} + \cdots + \binom{x-1}{k} = \binom{x+1}{l} + \cdots + \binom{y-1}{l}$ with given positive integers k and l and solved the cases $1 \leq k, l \leq 3$ completely.

Berczes, Liptai and Pink [5] considered a sequence defined by a binary recurrence $R_{n+1} = AR_n + BR_{n-1}$ with $A, B \neq 0$ and $|R_0| + |R_1| > 0$ and shown that if $A^2 + 4B > 0$ and $(A, B) \neq (0,1)$, no sequence balancing number exists in the above sequence $\{R_n\}_{n=1}^{\infty}$.

The definition of balancing numbers involves balancing sums of natural numbers. After the introduction of balancing numbers, Behera and Panda [3] considered the problem of balancing products of natural numbers. They called a positive integer n, a product balancing number if the Diophantine equation $1 \cdot 2 \cdot \cdots \cdot (n-1) = (n+1) \cdot \cdots \cdot (n+r)$ holds for some natural number r. They identified 7 as the first product balancing number, but couldn't find a second one. Subsequently, Szakács [48] proved that if n is a product balancing number then none of $(n+1),(n+2),\cdots,(n+r)$ is a prime and that no product balancing number other than 7 exists. He also proved the nonexistence of any product cobalancing number, that is, the Diophantine equation $1\cdot 2\cdot\ \cdots\ \cdot n=(n+1)\cdot\cdots\cdot(n+r)$ has no solution. However, he used the names multiplying balancing and multiplying cobalancing numbers in place of product balancing and product cobalancing numbers respectively.

Szakács [48] also defined a (k,l)-power multiplying balancing number as positive integers n satisfying the Diophantine equation $1^k\cdot 2^k\cdot\cdots\cdot(n-1)^k=(n+1)^l\cdot(n+2)^l\cdot\cdots\cdot(n+r)^l$ for some natural number r and proved that only one (k,l)-power multiplying balancing number corresponding to $k=l$ exists and is precisely $n=7$.

Cohn [8] investigated perfect squares in Fibonacci and Lucas sequence and showed that $L_n=x^2$ for $n=1,3$ and $F_n=x^2$ for $n=0,\ 1,\ 2\ ,12$. Subsequently, while searching for perfect squares in the balancing sequence, Panda [36] proved that there is no perfect square in the balancing sequence other than 1 by showing that $x=1,y=3$ is only positive solution of the Diophantine equation $8x^4+1=y^2$.

A perfect number is a natural numbers which is equal to the sum of its positive proper divisors. These numbers are very scarce and till date only 48 numbers are known. Thus, the chance of their adequacy in any number sequence is very less. While searching triangular numbers in the Pell sequence, Mc Daniel [28] proved that the only such number is $P_1=1$. Since every even perfect number is triangular, Mc Danial's finding is sufficient to establish the fact that there is no even perfect number in the Pell sequence. So far as the sequence of balancing numbers is concerned, Panda and Davala [40] managed to find one perfect number $B_2=6$ and further proved that no other balancing number is perfect.

A Diophantine n-tuple is a set $\{x_1,x_2,\cdots x_n\}$ of positive numbers such that the product of any two of them increased by 1 is a perfect square. Diophantus was first to introduce the concept of such quadruples by providing the example of the set

$\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{16}, \frac{105}{16}\right\}$ [30]. The first Diophantine quadruple $\{1,3,8,120\}$ in positive integers was obtained by Fermat. Later on, Baker and Davenport [2] proved that Fermat's set can't be extended to a Diophantine quintuple. They also conjectured that there no Diophantine quintuple exists.

Fuchs, Luca and Szalay [16] modified the concept of Diophantine n-tuple. They considered the problem of finding three integers a, b, c belonging to some integer sequence $\psi = \{a_n\}_{n=1}^{\infty}$ such that all $ab+1, ac+1, bc+1$ are members of ψ. Alp, Irmak and Szalay [1] proved the absence of any such triples in the balancing sequence.

Modular periodicity is an important aspect of any integer sequence. Wall [51] studied the periodicity of Fibonacci sequence modulo arbitrary natural numbers. He proved that the Fibonacci sequence modulo any positive integer m forms a simple periodic sequence. He further conjectured that there may be primes p such that the period of the Fibonacci sequence modulo p is equal to the period of the sequence modulo p^2. Elsenhans and Jahnel [13] extended this search for prime up to 10^{14}, but couldn't find any such prime. Niederreiter [31] proved that the Fibonacci sequence is uniformly distributed modulo m for $m = 5^k, k = 1,2,\cdots$.

Recently, Panda and Rout [39] studied the periodicity of balancing sequence and proved that the sequence of balancing numbers modulo any natural number m is periodic and $\pi(n)$, the period of the balancing sequence modulo n, is a divisibility sequence. They could not find any explicit formula for $\pi(n)$; however they managed to provide the value of $\pi(n)$ when n is a member of certain integer sequences, for example, the Pell sequence, the associated Pell sequence etc. They also showed that $\pi(2^k) = 2^k, k = 1, 2, \cdots$ and found three primes $13, 31$ and 1546463 such that the period of the balancing sequence modulo any of these three primes is equal to the period modulo its square. Subsequently, Rout, Davala and Panda [44] proved that the balancing sequence is stable for primes $p \equiv -1, -3 \pmod 8$ and not stable for primes $p \equiv 3 \pmod 8$.

There are numerous problems associated with any integer sequence. Kovacs, Liptai and Olajas [22] considered the problem of expressing balancing numbers as product of consecutive integers. They proved that the equation $B_n = x(x+1)\cdots(x+k-1)$ has only finitely many solutions for $k \geq 2$ and obtained all solutions for $k = 2, 3, 4$. Subsequently, Tengely [50] proved that the above Diophantine equation has no solution for $k = 5$.

Liptai [24,25] searched balancing numbers in Fibonacci and Lucas sequence and proved that there is no balancing number in the Fibonacci and Lucas sequence other than 1. Subsequently, Szalay [49] also proved the same result by converting the pair of Pell's equation $x^2 - 8y^2 = 1$ and $x^2 - 5y^2 = \pm 4$ into a family of Thue equations.

Cerin [7] studied certain geometric properties of triangles such as area properties and orthology and paralogy of triangles with coordinates from the Fibonacci, Lucas, Pell and Lucas-Pell sequence. Davala and Panda [10] extended this study to polygons in the plane. They explored areas of polygons and developed certain families of orthologic and paralogic triangles.

Dash and Ota [9] generalized the concept of balancing number in another innovative way to defining t-balancing numbers. They call a natural number n a t-balancing number if $1 + 2 + \cdots + n = (n+1+t) + (n+2+t) + \cdots + (n+r+t)$ holds for some r. These numbers coincide with balancing numbers when $t = 0$ and enjoy certain properties analogous to balancing numbers.

Keskin and Karath [19] studied some other important aspects of balancing numbers. They showed there is no Pythagorean triple with coordinates as balancing numbers. They further proved that the product of two balancing numbers other than 1 is not a balancing number.

Ray, Dila and Patel [42] studied the application of balancing and Lucas-balancing numbers to a cryptosystem involving hill cyphers. Their method is based on the application of hill cipher using recurrence relation of balancing Q-matrix.

The contents of this thesis have been divided into seven chapters. In Chapter 2, we present some known literature review required for the development of subsequent chapters. We try to keep this chapter little elaborate to make this work self-contained. In the subsequent chapters excepting the last one, we study several interesting generalizations of balancing and cobalancing sequence, while in the last chapter we establish the involvement of balancing-like sequences in a statistical Diophantine problem.

Now we describe briefly the different types of generalizations of balancing and cobalancing numbers studied in this thesis.

The balancing numbers are defined as the natural numbers n satisfying the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + m$ for some natural number m. There are certain values of n such that the left and right hand side of the above equation are almost equal. For example, one may be interested to explore those n satisfying $|[1 + 2 + \cdots + (n-1)] - [(n+1) + (n+2) + \cdots + m]| = 1$; we call such a n's as almost balancing numbers. It is obvious that the definition results in two types of almost balancing numbers. We carry out a detailed study of such numbers in Chapter 3.

After going through the generalization from balancing numbers to almost balancing numbers, a natural question may strike to one's mind: "Is it possible to generalize cobalancing numbers to almost cobalancing numbers?" In Chapter 4, we answer this question in affirmative. The method of generalization is similar to that discussed in the last para.

The balancing and cobalancing numbers are defined by means of Diophantine equations that involve balancing of sums of natural numbers. On the other hand, the almost balancing and cobalancing numbers are defined by maintaining a difference 1 in the left hand and right hand sides in the defining equations of balancing and cobalancing numbers respectively. While generalizing the balancing sequence to balancing-like sequences, the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$ of the balancing sequence has been generalized by Panda and Rout [37] to $x_{n+1} = Ax_n - x_{n-1}$ without disturbing the initial values and allowing A being any natural number greater than 2. Now question arises, "How can one generalize the balancing-like sequences to almost balancing-like sequences in the line of generalization of the balancing sequence to the balancing-like sequence?" Since balancing-like sequences do not have defining equations like the balancing sequence, one needs a different means of generalization. It is well-known a natural number x is a balancing or cobalancing number according as $8x^2 + 1$ or $8x^2 + 8x + 1$ is a perfect square. Further, a natural number x is an almost balancing or almost cobalancing number according as $8(x^2 \pm 1) + 1$ or $8(x^2 + x \pm 1) + 1$ is a perfect square. Since for fixed A and with $D = (A^2 - 4)/4$, x is a balancing-like number if and only if $Dx^2 + 1$ is a perfect square, we call x an almost balancing-like number if and only if $D(x^2 \pm 1) + 1$ is a perfect square. Chapter 4 is entirely devoted to the study of almost balancing-like sequences.

A balancing number is such that if it is deleted from certain string of consecutive natural numbers starting with 1, the sum to the left of this deleted number is equal to the sum to its right. A generalization is possible by considering a circular necklace of consecutive natural numbers equally spaced as beads. If by removing two numbers corresponding to a chord joining the beads k and $n\ (> k)$, the sum of numbers on both arcs is same, we call n a k-circular balancing number. We employ Chapter 6 for an extensive study of circular balancing numbers.

After going through several generalizations of the balancing sequence, a question may strike to someone's mind, "Is there any relation of any such sequence with other areas of mathematical science? In Chapter 7, we answer this question in affirmative by studying a statistical Diophantine problem associated with balancing-like sequences.

Chapter 2

Preliminaries

In this chapter, we recall some known theories, definitions and results which are necessary for this work to become self-contained. Some contents of this chapter are necessary for the development of subsequent chapters. We shall keep on referring back to this chapter as and when necessary without further reference.

2.1 Diophantine equation

A Diophantine equation is an algebraic equation in one or more unknowns whose integer solutions are sought. The Greek number theorist Diophantus who is known for his book *Arithmetica* first studied these types of equations.

The Diophantine equation $x^2 + k = y^3$ was first studied by Bachet in 1621 and has played a fundamental role in the development of number theory. When $k = 2$, the only integral solutions to this equation are given by $y = 3,\ x = \pm 5$. It is known that the equation has no integral solution for many values of k.

The Pythagorean equation $x^2 + y^2 = z^2$ is a most popular Diophantine equation and the positive integral triplet (x, y, z) satisfying the above equation is called a Pythagorean triple. The existence of infinitude of its solutions is well-known.

The most famous Diophantine equation is due to Fermat **(**1607-1665**)** known as the Fermat's last theorem (FLT) which states that the Diophantine equation $x^n + y^n = z^n$ has no solution in positive integers if $n > 2$. In 1637, Fermat wrote on the margin of a copy of the book Arithmetica: "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two likes powers. I have truly found a wonderful proof of this result which this margin is too narrow to contain." Many famous mathematicians tried in vain for about three and half centuries but could not provide a proof. Finally, the British mathematician Andrew Wiles gave a proof of this theorem in 1993 in which an error was detected. He corrected the proof in 1994 and finally published in 1995.

As a generalization of Fermat's last theorem, Andrew Beal, a banker and an amateur mathematician formulated a conjecture in 1993. It states that the Diophantine equation $x^m + y^n = z^r$ where x, y, z, m, n and r are positive integers and m, n and r are all greater than 2 then x, y and z must have a common prime factor. In 1997, Beal announced a monetary prize for a peer-reviewed proof of this conjecture or a counter example. The value of the prize has been increased several times and its current value is 1 million dollar.

In the theory of Diophantine equation, there is another important conjecture called Catalan's conjecture which states that the only solution in natural numbers of the Diophantine equation $x^a - y^b = 1$ for $x, y, a, b > 1$ is $x = 3, y = 2, a = 2, b = 3$. The conjecture was proved by Preda Mihailescu in the year 2000.

2.2 Pell's Equation

A Diophantine equation of the form $x^2 - Dy^2 = 1$, where D is a non-square, is known as Pell's equation. The English mathematician John Pell (1611–1685), for whom this equations is known as Pell's equation, has nothing to do with this equation. Euler (1707–1783) by mistake, attributed to Pell a solution method that had in fact been found by the English mathematician William Brouncker (1620–1684) in response to a challenge by Fermat (1601–1665); but attempts to change the terminology introduced by Euler have always proved futile.

The Pell's equation $x^2 - Dy^2 = 1$ can be written as

$$\left(x + y\sqrt{D}\right)\left(x - y\sqrt{D}\right) = 1.$$

Thus, finding solutions of the Pell's equation reduces to obtain non-trivial units of the ring $\mathbb{Z}[\sqrt{D}]$ of norm 1. Here, the norm $\mathbb{Z}[\sqrt{D}]^* \to \{\pm 1\}$ between unit groups multiplies each unit by its conjugates and the units ± 1 of $\mathbb{Z}[\sqrt{D}]$ are considered trivial. This formulation implies that if one knows a solution to Pell's equation, he can find infinitely many. More precisely, if the solutions are ordered by magnitude, then the n^{th} solution (x_n, y_n) can be expressed in terms of the first nontrivial positive solution (x_1, y_1) as

$$x_n + y_n\sqrt{D} = (\, x_1 + y_1\sqrt{D})^n, n = 1,2, \cdots.$$

Instead of $(\, x_n, y_n \,)$, it is customary to call $x_n + y_n\sqrt{D}$ as the n^{th} solution of the Pell's equation. Accordingly, the first solution $x_1 + y_1\sqrt{D}$ is called the fundamental solution of the Pell's equation, and solving the Pell's equation means finding (x_1, y_1) for any given D.

The Diophantine equation $x^2 - Dy^2 = N$, where D is a positive non-square integer and $N \notin \{0,1\}$ is any integer, is known as a generalized Pell's equation. This equation has either no solution or has infinite numbers of solutions. Further, the solutions constitute a single class or may partition in multiple classes. The solutions of any class can be obtained from

$$x_n' + y_n'\sqrt{D} = \left(x_1 + y_1\sqrt{D}\right)\left(u_1 + v_1\sqrt{D}\right)^{n-1}, n = 1,2, \cdots \tag{2.1}$$

where $u_1 + v_1\sqrt{D}$ is the fundamental solution of the equation $x^2 - Dy^2 = 1$ and $x_1 + y_1\sqrt{D}$ is a fundamental solution of $x^2 - Dy^2 = N$. It is easy to see that two solutions of $x^2 - Dy^2 = N$ are in same class if and only if their ratio is a solution of $x^2 - Dy^2 = 1$.

The following theorem determines the bounds for the fundamental solutions of a generalized Pell's equation [29].

2.2.1 Theorem. *Let $n > 1$ and let $x + y\sqrt{D}$ be a fundamental solution* of $x^2 - Dy^2 = N$. *If $a + b\sqrt{D}$ is a fundamental solution* of $x^2 - Dy^2 = 1$, *then*

$$0 < |x| \leq \sqrt{\frac{(a+1)N}{2}}, \;\; 0 \leq y \leq \frac{b\sqrt{N}}{\sqrt{2(a+1)}}.$$

2.3 Recurrence relation

To understand a sequence $\{a_n\}$ completely, it is necessary to write its n^{th} term as a function of n. For example $a_n = \frac{2^n}{n!}, n = 1,2, \ldots$ results in the sequence 2, 2, $\frac{4}{3}$, $\frac{2}{3}$, $\frac{4}{15}, \ldots$. However, some sequences are better understood by means of a dependence relation of each term on some of its previous terms with specification of certain initial terms. Such sequences are known as recurrence sequences and the dependence relation of the current terms on the previous terms is known as a recurrence relation, or simply a recurrence.

A k^{th} order linear recurrence relation with constant coefficient is an equation of the form

$$a_{n+1} = c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + c_{k-1} a_{n-k+1} + f(n), n \geq k$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants, $c_{k-1} \neq 0$. When $f(n) = 0$, the corresponding recurrence is called homogeneous, otherwise it is called nonhomogeneous. To explore the sequence $\{a_n\}$ completely, the values of the first k terms of the sequence need to be specified. They are called the initial values of the recurrence relation and allow one to compute a_n, for each $n \geq k$.

For any k^{th} order homogenous recurrence relation $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_{k-1} a_{n-k+1} = 0$ with given initial values $a_0, a_1, \cdots, a_{k-1}$, there is an associated equation

$$\alpha^k + c_1 \alpha^{k-1} + \cdots + c_{k-1} \alpha + c_k = 0$$

called the characteristic equation and its roots are known as the characteristic roots. If the characteristic roots $\alpha_1, \alpha_2, \cdots, \alpha_k$ are all real and distinct then the general solution of the recurrence is given by

$$a_n = A_1 \alpha_1^n + A_2 \alpha_2^n + \cdots + A_k \alpha_k^n \tag{2.2}$$

and a closed form (commonly known as the Binet form) of the recurrence which can be obtained by finding the values of $A_1, A_2, \cdots, A_k$ using the initial values and substituting in (2.2). In particular, the characteristic equation of a linear homogeneous recurrence relation of second order (also commonly known as a binary recurrence) is of the form

$\alpha^2 + c_1\alpha + c_2 = 0$ which has two roots α_1 and α_2. If both the roots are distinct and real then the general solution of the binary recurrence is given by

$$a_n = A_1\alpha_1^n + A_2\alpha_2^n.$$

However, in case of equal roots, that is $\alpha_1 = \alpha_2 = \alpha$, the general solution is given by

$$a_n = (A_1 + A_2 n)\alpha^n.$$

In case of complex conjugate roots say $\alpha_1 = re^{i\theta},\ \alpha_2 = re^{-i\theta}$, the solution is expressed as

$$a_n = (A_1 cosn\theta + A_2 sinn\theta)r^n.$$

In all the above cases, the two initial values determines the unknowns A_1 and A_2.

2.4 Triangular numbers

A triangular number is a figurate number that can be represented by an equilateral triangular arrangement of points equally spaced. The n^{th} triangular number is denoted by T_n and is equal to $\frac{n(n+1)}{2}$. These numbers appear in Row 3 of the Pascal's triangle.

A number which is simultaneously triangular and square is known as a square triangular number. There is an infinitude of square triangular numbers and these numbers can be easily be calculated by means of a binary recurrence $ST_{n+1} = 34ST_n - ST_{n-1} + 2$ with initial values $ST_0 = 0$ and $ST_1 = 1$ [41, p.22], where ST_n denoted the n^{th} square triangular number. Square triangular numbers are squares of balancing numbers [3].

A pronic number (also known as oblong number) is a figurate number that can be represented by a rectangular arrangement of points equally spaced such that the length is just one more than the breadth. A number which is simultaneously pronic and triangular is known as a pronic triangular number. There are infinitely many pronic triangular numbers and these numbers can also be calculated using the binary recurrence $PT_{n+1} = 34PT_n - PT_{n-1} + 6$ with initial values $PT_0 = 0$ and $PT_1 = 6$, where PT_n denotes the n^{th} pronic triangular number. Pronic triangular numbers are very closely related to cobalancing numbers [41, p.35]. These numbers are also related to the balancing numbers in the sense that the product of any two consecutive balancing numbers is a pronic number and all the pronic triangular numbers are of this form.

2.5 Fibonacci numbers

The Fibonacci sequence is defined by means of the binary recurrence, $F_{n+1} = F_n + F_{n-1}, n \geq 2$ with initial values $F_0 = 0$ and $F_1 = 1$. The n^{th} Fibonacci number F_n can be expressed explicitly using Binet's formula as

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. There are many Fibonacci identities. The following is a list of some important ones.

- $F_{-n} = (-1)^{n+1} F_n$
- $F_{n-1}F_{n+1} - F_n^2 = (-1)^n, n \geq 1$ (Cassini formula)
- $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$
- $\frac{F_{n+1}}{F_n} \to \alpha \; as \; n \to \infty$, where $\alpha = \frac{1+\sqrt{5}}{2}$
- $\sum_{i=1}^{n} F_i = F_{n+2} - 1$
- $F_n = F_m F_{n-m+1} + F_{m-1} F_{n-m}$
- $F_m | F_n$ if and only if $m | n$

2.6 Balancing numbers

According to Behera and Panda [3], a natural number B is a balancing number with balancer R if the pair (B, R) satisfies the Diophantine equation

$$1 + 2 + \cdots + (B - 1) = (B + 1) + (B + 2) + \cdots + (B + R).$$

It is well-known that a positive integer B is a balancing number if and only if B^2 is a triangular number, or equivalently $8B^2 + 1$ is a perfect square and the positive square root of $8B^2 + 1$ is called as the Lucas-balancing number.

The n^{th} balancing and Lucas-balancing numbers are denoted by B_n and C_n respectively and their Binet forms are given by

$$B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}}, \;\; C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2}$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$. The balancing and Lucas-balancing numbers are solutions of a single binary recurrence with different initial values. In particular, $B_0 = 0, B_1 = 1, C_0 = 1, C_1 = 3$ and

$$B_{n+1} = 6B_n - B_{n-1}\,,$$

and

$$C_{n+1} = 6C_n - C_{n-1}.$$

Balancing and Lucas-balancing numbers share some interesting properties. In many identities, Lucas-balancing numbers are associated with balancing numbers the way Lucas numbers are associated with Fibonacci numbers. The following are some important identities involving balancing and/or Lucas-balancing numbers.

- $B_{-n} = -B_n,\; C_{-n} = C_n$
- $B_{n+1}.B_{n-1} = {B_n}^2 - 1$
- $B_{m+n+1} = B_{m+1}\, B_{n+1} - B_m\, B_n$
- $B_{2n-1} = B_n^2 - B_{n-1}^2$
- $B_{2n} = B_n(B_{n+1} - B_{n-1})$
- $B_{m+n} = B_m C_n + C_m B_n$
- $C_{m+n} = C_m C_n + 8B_m B_n$
- $B_1 + B_3 + \cdots + B_{2n-1} = B_n^2$
- $B_2 + B_4 + \cdots + B_{2n} = B_n B_{n+1}$
- $B_m | B_n$ if and only if $m|n$

2.7 Cobalancing numbers

The cobalancing numbers and cobalancers are solutions of a Diophantine equation similar to that of the defining equation of balancing numbers and balancers. As defined by Panda and Ray [32], a natural number b is a cobalancing number with cobalancer r if the pair (b, r) satisfies the Diophantine equation

$$1 + 2 + \cdots + b = (b + 1) + (b + 2) + \cdots + (b + r).$$

The above definition suggests that a natural number b is a cobalancing number if and only if $8b^2 + 8b + 1$ is a perfect square [41, p.35] or equivalently, the pronic number $b(b + 1)$ is triangular. In other words, if N is a pronic triangular number then $\lfloor\sqrt{N}\rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor function, is a cobalancing number.

The cobalancing numbers can also be calculated using the binary recurrence

$$b_{n+1} = 6b_n - b_{n-1} + 2$$

with initial values $b_0 = b_1 = 0$. This recurrence is not much different from that of balancing and Lucas balancing numbers except the presence of 2 in the right hand side which makes it nonhomogeneous.

Balancing numbers and balancers are very closely related to cobalancing numbers and cobalancers. Panda and Ray [32] proved that every balancer is a cobalancing number and every cobalancer is a balancing number.

If b is a cobalancing number then the positive square root of $8b^2 + 8b + 1$ is called a Lucas-cobalancing number. The n^{th} Lucas-cobalancing number is denoted by c_n and these numbers are solutions of binary recurrence

$$c_{n+1} = 6c_n - c_{n-1}$$

with initial values $c_0 = c_1 = 1$. The Binet form for the cobalancing and Lucas-cobalancing numbers are given by

$$b_n = \frac{\alpha_1^{2n-1} - \alpha_2^{2n-1}}{4\sqrt{2}} - \frac{1}{2}, \quad c_n = \frac{\alpha_1^{2n-1} + \alpha_2^{2n-1}}{2},$$

where $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

The cobalancing numbers enjoy many beautiful identities. They also share many identities with balancing numbers. The following are some of these identities.

- $b_{n+1} = 3b_n + \sqrt{8b_n^2 + 8b_n + 1} + 1$
- $b_{n-1} = 3b_n - \sqrt{8b_n^2 + 8b_n + 1} + 1$
- $(b_n - 1)^2 = b_{n-1}b_{n+1} + 1$
- $2(B_1 + B_2 + \cdots + B_n) = b_{n+1}$
- $b_{m+n} = b_m + B_m b_{n+1} - B_{m-1} b_n$
- $b_{2n+1} = (B_{n+1} + 1)b_{n+1} - B_n b_n.$

2.8 Pell and associated Pell numbers

The Pell sequence $\{P_n\}$ and the associated Pell sequence $\{Q_n\}$ are defined recursively as

$$P_{n+1} = 2P_n + P_{n-1}$$

and

$$Q_{n+1} = 2Q_n + Q_{n-1}$$

with initial values $P_0 = 0, P_1 = 1, Q_0 = Q_1 = 1$. The importance of Pell and associated Pell sequence lies in the fact that the ratios $\frac{Q_n}{P_n}$ appear as successive convergents in the continued fraction expansion of $\sqrt{2}$. These sequences are solutions of the Pell's equations $y^2 - 2x^2 = \pm 1$. The Binet forms of these sequences are given by

$$P_n = \frac{\alpha_1^n - \alpha_2^n}{2\sqrt{2}},\ Q_n = \frac{\alpha_1^n + \alpha_2^n}{2}$$

where α_1 and α_2 are as defined in Section 2.7. There are many import identities involving Pell and associated Pell numbers. In the following we list some of these.

- $P_{-n} = (-1)^{n+1}P_n\ ,\ \ Q_{-n} = (-1)^n Q_n$
- $P_{m+n} = P_m P_{n+1} + P_{m-1} P_n$

- $P_{m+n} = 2P_m Q_n - (-1)^n P_{m-n}$
- $Q_n^2 = 2P_n^2 + (-1)^n$
- $Q_{2n} = 2Q_n^2 - (-1)^n$
- $P_{n+1}P_{n-1} - P_n^2 = (-1)^n$.

The Pell and associated Pell sequences are in close association with the sequences of balancing, cobalancing, Lucas-balancing and Lucas-cobalancing numbers. We present below some important relationships.

- $B_n = P_n Q_n$
- $P_{2n} = 2B_n$
- $Q_{2n} = C_n$ and $Q_{2n-1} = c_n$
- $P_1 + P_2 + \cdots + P_{2n-1} = B_n + b_n$
- $P_{n+1}^2 - P_{n-1}^2 = C_n + c_n$.

2.9 Gap balancing numbers

In the definition of balancing numbers, a number is deleted (hence a gap is maintained) from the list of first m natural numbers so that the sum of the numbers to the left of this deleted number is equal to the sum to right. In case of cobalancing numbers [32], no such gap is created. Recently, Panda and Rout [38, 43] considered deleting two or more consecutive numbers, not from either end, from the list of first m natural numbers so that the sum of the numbers to the left of these deleted numbers is equal to the sum to their right. This consideration results in the introduction of gap balancing numbers. In this context, the cobalancing numbers and balancing numbers are identified as 0-gap and 1-gap balancing numbers respectively.

Panda and Rout [43] defined k-gap balancing numbers as follows:

2.9.1 Definition. Let k be an odd natural number. A natural number n is called a k-gap balancing number if

$$1+2+\cdots+\left(n-\frac{k+1}{2}\right)=\left(n+\frac{k+1}{2}\right)+\left(n+\frac{k+3}{2}\right)+\cdots+(n+r)$$

for some natural number r, which is known as the k-gap balancer corresponding to n. Similarly for an even natural number k, if

$$1+2+\cdots+\left(n-\frac{k}{2}\right)=\left(n+\frac{k}{2}+1\right)+\left(n+\frac{k}{2}+2\right)+\cdots+(n+r)$$

for some natural number r, $2n+1$ is called a k-gap balancing number and r is the corresponding k-gap balancer.

Panda and Rout [38] showed that there are two classes of 2-gap balancing numbers expressible in terms of balancing and Lucas-balancing numbers as $6B_n-C_n$ and $6B_n+C_n$ respectively.

So far as the k-gap balancing numbers are concerned, Panda and Rout [43] established two or more classes of these numbers for $k=3,4,5$.

2.10 Balancing-like sequences

The balancing numbers are solutions of a binary recurrence $B_{n+1}=6B_n-B_{n-1}$ with initial values $B_0=0$, $B_1=1$. Panda and Rout [37] studied a natural generalization of the balancing sequence by considering sequences which are solutions of the class of binary recurrences $x_{n+1}=Ax_n-x_{n-1}$ with initial values $x_0=0, x_1=1$, where $A>2$ is any arbitrary positive integer. These sequences are subsequently known as balancing-like sequences.

Though, the case $A=2$ is excluded from the definition of balancing-like sequences, it is important to note that the binary recurrence $x_{n+1}=2x_n-x_{n-1}$ with initial values $x_0=0, x_1=1$ represents the natural numbers. Hence, the balancing-like sequences are thought of as generalizations of the sequence of natural numbers and are sometimes termed as natural sequences. The balancing-like sequence corresponding to $A = 3$ is the sequence of even indexed Fibonacci numbers and the balancing-like sequence corresponding to $A = 6$ is the sequence of balancing numbers.

For any fixed positive integer $A > 2$ and for each $n, Dx_n^2 + 1$, where $D = (A^2 - 4)/4$, is a perfect rational square and the sequence $\{y_n\}_{n=1}^{\infty}$ defined by

$$y_n = \sqrt{Dx_n^2 + 1}$$

is called the Lucas-balancing-like sequence corresponding to the balancing-like sequence $\{x_n\}_{n=1}^{\infty}$. The Lucas-balancing-like sequences are associated with their balancing-like sequences, the way the Lucas-balancing sequence is associated with the balancing sequence. The balancing-like and the Lucas-balancing like sequences enjoy properties analogues to those listed in Section 2.6.

2.11 Almost and nearly Pythagorean triples

A Pythagorean triple (x, y, z) is a solution of the Pythagorean equation $x^2 + y^2 = z^2$ in positive integers and is called primitive if the greatest common divisor of x and y is 1. Every Pythagorean triple is an integral multiple of some primitive Pythagorean triple. The list of all primitive Pythagorean triples is given by

$$x = r^2 - s^2, y = 2rs, z = r^2 + s^2$$

where r and s are relatively prime positive integers of opposite parity.

Orrin Frink [15] defined almost Pythagorean triples (x, y, z) as solutions of the Diophantine equation $x^2 + y^2 = z^2 + 1$. Similarly, he defined nearly Pythagorean triples as solutions of $x^2 + y^2 = z^2 - 1$. He established a relationship between almost Pythagorean triples and nearly Pythagorean triples. He also proved that, if (p, q, r) is an almost Pythagorean triple, then $(2pr,\ 2qr,\ 2r^2 + 1)$ is a nearly Pythagorean triple and if (p, q, r) is a nearly Pythagorean triple, then $(2p^2 + 1,\ 2pq,\ 2pr)$ is an almost Pythagorean triple.

Chapter 3

Almost Balancing Numbers

3.1 Introduction

As discussed in Chapter 2, balancing numbers n and balancers r are solutions of the Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$. It is well-known that the square of a balancing number is a triangular number and there is a one-to-one correspondence between square triangular numbers and balancing numbers. In 1999, Subramaniam [47] introduced the concept of almost square triangular numbers (triangular numbers that differ from a square by unity) and establish their links with square triangular numbers. In this chapter, with a little modification in the defining equation for balancing numbers, we introduce the concept of almost balancing numbers and show that these numbers are associated with almost square triangular numbers the way balancing numbers are associated with square triangular numbers. We also explore certain important aspects of these numbers and examine their relationships with balancing and Lucas-balancing numbers.

3.2 Definitions and preliminaries

This section is devoted to the definitions of the two types of almost balancing numbers. Some examples of these numbers are also provided.

3.2.1 Definition. We call a natural number n an *almost balancing number* if

$$|\{(n+1) + (n+2) + \cdots + (n+r)\} - \{1 + 2 + \ldots + (n-1)\}| = 1$$

for some natural number r, which we call the *almost balancer* corresponding to n.

It is clear from the above definition that the almost balancing numbers can be partitioned in two classes. If

$$\{(n+1) + (n+2) + \cdots + (n+r)\} - \{1+2+\cdots+(n-1)\} = 1,$$

we call n, an *almost balancing number of first kind* while if

$$\{(n+1) + (n+2) + \cdots + (n+r)\} - \{1+2+\cdots+(n-1)\} = -1,$$

we call n, an *almost balancing number of second kind*. In the former case we call r, an almost balancer of first kind, while in the latter case we call r, an almost balancer of second kind.

For the sake of simplicity, we call almost balancing numbers of first kind as A_1-*balancing numbers* and almost balancers of first kind as A_1-*balancers*. Similarly, we call almost balancing number of second kind as A_2-*balancing numbers* and almost balancers of second kind as A_2-*balancers*.

Observe that, if n is an A_1-balancing number with A_1-balancer r, then

$$\frac{n(n-1)}{2} + 1 = \frac{(n+r)(n+r+1)}{2} - \frac{n(n+1)}{2}$$

or equivalently,

$$n^2 + 1 = \frac{(n+r)(n+r+1)}{2}$$

showing that n^2+1 is a triangular number. Equivalently, a natural number n is an A_1-balancing number if and only if $8n^2+9$ is a perfect square. Similarly, if n is an A_2-balancing number with A_2-balancer r, then

$$n^2 - 1 = \frac{(n+r)(n+r+1)}{2},$$

that is, n^2-1 is a triangular number. Thus, a natural number $n > 2$ is an A_2-balancing number if and only if $8n^2-7$ is a perfect square.

3.2.2 Examples. Since $8 \cdot 3^2 + 9 = 9^2$ and $8 \cdot 18^2 + 9 = 51^2$, 3 and 18 are A_1-balancing numbers. Similarly, 4 and 11 are A_2-balancing numbers since $8 \cdot 4^2 - 7$ and $8 \cdot 11^2 - 7$ are perfect squares.

3.3 Listing all almost balancing numbers

If $x > 2$ be an A_1-balancing number, then there exists a natural number y such that $8x^2 + 9 = y^2$. We claim that $x \equiv 0(\text{mod } 3)$. Observe that if $x \equiv \pm 1(\text{mod } 3)$ then $8x^2 + 9 \equiv -1(\text{mod } 3)$; but -1 is a quadratic nonresidue modulo 3, a contradiction to $y^2 \equiv -1(\text{mod } 3)$. Thus, $x \equiv 0$ (mod 3) and consequently $y \equiv 0(\text{mod } 3)$. Let u and v be natural numbers such that $x = 3u$ and $y = 3v$. Substituting these values in $8x^2 + 9 = y^2$ we get $8u^2 + 1 = v^2$, implying that u^2 is a triangular number, hence u is a balancing number [3] and v is the corresponding Lucas-balancing number. Thus, the solutions of $8x^2 + 9 = y^2$ is given by $x = 3B_n$ and $y = 3C_n, n = 1,2, ...$

.

We can summarize the above discussion as follows.

3.3.1 Theorem. *The solutions of the Diophantine equation* $8x^2 + 9 = y^2$ *in positive integers are given by* $x = 3B_n$ *and* $y = 3C_n$, $n = 1,2,\cdots$. *In particular, the* A_1-*balancing numbers are given by* $3B_n, n = 1,2,\cdots$.

We next explore all the positive integers x such that $8x^2 - 7$ is a perfect square. Equivalently, this requires solving the generalized Pell's equation $y^2 - 8x^2 = -7$. The fundamental solution of $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$ and there are two classes of fundamental solutions of $y^2 - 8x^2 = -7$, namely $1 + \sqrt{8}$ and $5 + 2\sqrt{8}$. Now, the solutions corresponding to the first class are given by

$$y_n + \sqrt{8}x_n = (1 + \sqrt{8})(3 + \sqrt{8})^{n-1}, \qquad n = 1,2, ... \tag{3.1}$$

and the solutions in the second class are

$$y_n' + \sqrt{8}x_n' = (5 + 2\sqrt{8})(3 + \sqrt{8})^{n-1}, \qquad n = 1,2,\cdots. \tag{3.2}$$

Solving (3.1) and (3.2) for x_n, y_n, x_n' and $y_n{}'$, we get

$$x_n = \frac{\left(3+\sqrt{8}\right)^n - \left(3-\sqrt{8}\right)^n}{2\sqrt{8}} - 2\cdot\frac{\left(3+\sqrt{8}\right)^{n-1} - \left(3-\sqrt{8}\right)^{n-1}}{2\sqrt{8}},$$

$$y_n = \frac{\left(3+\sqrt{8}\right)^n + \left(3-\sqrt{8}\right)^n}{2} - 2\cdot\frac{\left(3+\sqrt{8}\right)^{n-1} + \left(3-\sqrt{8}\right)^{n-1}}{2},$$

$$x_n{}' = 2\cdot\frac{\left(3+\sqrt{8}\right)^n - \left(3-\sqrt{8}\right)^n}{2\sqrt{8}} - \frac{\left(3+\sqrt{8}\right)^{n-1} - \left(3-\sqrt{8}\right)^{n-1}}{2\sqrt{8}},$$

$$y_n{}' = 2\cdot\frac{\left(3+\sqrt{8}\right)^n + \left(3-\sqrt{8}\right)^n}{2} - \frac{\left(3+\sqrt{8}\right)^{n-1} + \left(3-\sqrt{8}\right)^{n-1}}{2}.$$

Comparing the expressions for $x_n,\ y_n,\ x_n'$ and $y_n{}'$ with the Binet forms of balancing and Lucas –balancing numbers, we finally get

$$x_n = B_n - 2B_{n-1},\ y_n = C_n - 2C_{n-1},$$

$$x_n' = 2B_n - B_{n-1},\ y_n' = 2C_n - C_{n-1}.$$

The above discussion can be summarized as follows.

3.3.2 Theorem. *The solutions of the Diophantine equation* $8x^2 - 7 = y^2$ *in positive integers partition into two classes. The first class of solutions is* $(x, y) = (B_n - 2B_{n-1}, C_n - 2C_{n-1})$ *and the second class is* $(x, y) = (2B_n - B_{n-1}, 2C_n - C_{n-1}),\ n = 1,2, \ldots$ *. In particular, all the of* A_2*-balancing numbers are given by* $\{B_n - 2B_{n-1},\ 2B_n - B_{n-1},\ n = 1,2, \ldots\}$.

3.4 Transformations from balancing to almost balancing and almost balancing to balancing numbers

We denote the n^{th} A_1-balancing number by U_n and n^{th} A_2-balancing number by V_n. By virtue of Theorem 3.3.1 and 3.3.2, $U_n = 3B_n$, $V_{2n-1} = B_n - 2B_{n-1}$ and $V_{2n} = 2B_n - B_{n-1}$. The objective in this section is to find functions transforming balancing numbers to almost balancing numbers, almost balancing numbers to balancing numbers and almost balancing numbers to almost balancing numbers.

The transformations from balancing numbers to A_1-balancing numbers, A_1-balancing numbers to balancing numbers and A_1-balancing numbers to A_1-balancing numbers are obvious. For example, if x is any balancing number then $3x$ is an A_1-balancing number and if x is any A_1-balancing number then $x/3$ is a balancing number. Again, if x is a balancing number then the next one is $3x + \sqrt{8x^2+1}$. Thus, if y is a A_1-balancing number then the next one is $3y + \sqrt{8y^2+9}$ since in this case, $y = 3x$ where x is a balancing number and

$$\begin{aligned} 3y + \sqrt{8y^2+9} &= 9x + \sqrt{72x^2+9} \\ &= 3\left(3x + \sqrt{8x^2+1}\right). \end{aligned}$$

The functions of interest are the ones that transform balancing numbers to A_2-balancing numbers. We have already proved that the odd ordered A_2-balancing numbers are of the form $B_n - 2B_{n-1}$, while the even order numbers are of the form $2B_n - B_{n-1}$. By virtue of the relation $B_{n-1} = 3B_n - C_n$ and $C_n = \sqrt{8B_n^2+1}$, the even and odd ordered A_2-balancing numbers can be expressed in the forms $-5B_n + 2C_n$ and $-B_n + C_n$ respectively.

The above discussion proves the following theorem.

3.4.1 Theorem. *If x is a balancing number then $\alpha(x) = -5x + 2\sqrt{8x^2+1}$ and $\beta(x) = -x + \sqrt{8x^2+1}$ are A_2-balancing numbers.*

We next consider the problem of finding functions that transform A_2-balancing numbers to balancing numbers. It is easy to check that the functions $\alpha(x)$ and $\beta(x)$ are strictly increasing in the domain $[1, \infty)$. Hence their inverses exist and are equal to

$$\alpha^{-1}(y) = \frac{5y + 2\sqrt{8y^2-7}}{7},\ \beta^{-1}(y) = \frac{y + \sqrt{8y^2-7}}{7}$$

respectively.

The above discussion leads to the following theorem.

3.4.2 Theorem. *If x is an odd ordered A_2-balancing number then $f(x) = \frac{5x+2\sqrt{8x^2-7}}{7}$ is a balancing number, while if x is an even ordered A_2-balancing number then $g(x) = \frac{x+\sqrt{8x^2-7}}{7}$ is a balancing number.*

3.5 Recurrence relations and Binet forms for almost balancing numbers

In the previous section, we have shown that the A_1-balancing numbers are given by $U_n = 3B_n, n = 1,2,\cdots$. Since the balancing numbers satisfy the recurrence relation $B_0 = 0,\ B_1 = 1$ and for $n \geq 1,\ B_{n+1} = 6B_n - B_{n-1}$, the A_1-balancing numbers satisfy $U_0 = 0, U_1 = 3$ and for $n \geq 1$,

$$U_{n+1} = 6U_n - U_{n-1}\ .$$

Using the Binet form of balancing numbers, we can easily write the Binet form of A_1-balancing numbers as

$$U_n = 3 \cdot \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}}, n = 1,2, \ldots$$

where, as usual, $\alpha_1 = 1 + \sqrt{2}$ and $\alpha_2 = 1 - \sqrt{2}$.

The A_2-balancing numbers, although directly related to balancing numbers, do not have a linear recurrence relation of second order like A_1-balancing numbers. Indeed, using the recurrence relations for balancing numbers, one can easily check that the A_2-balancing numbers satisfy the fourth order recurrence relation

$$V_{n+2} = 6V_n - V_{n-2}$$

with initial terms $V_0 = 1, V_1 = 1, V_2 = 2, V_3 = 4$. Indeed, we need not have to solve this recurrence to get the Binet form. Rather, using the Binet forms of balancing and Lucas-balancing numbers, we can obtain

$$V_{2n-1} = B_n - 2B_{n-1} = 2C_n - 5B_n$$

$$= \frac{(2\sqrt{2}+1)\alpha_1^{2(n-1)} - (2\sqrt{2}-1)\alpha_2^{2(n-1)}}{4\sqrt{2}}, n = 1,2, \ldots$$

and

$$V_{2n} = 2B_n - B_{n-1} = C_n - B_n$$

$$= \frac{(2\sqrt{2}-1)\alpha_1^{2n} + (2\sqrt{2}+1)\alpha_2^{2n}}{2}, n = 1,2, \dots .$$

There are some other interesting relations among almost balancing, balancing and Lucas-balancing numbers. The following theorem is important in this regard.

3.5.1 Theorem. *For any natural number n, $2V_{2n} - V_{2n-1} = U_n$ and consequently, $\frac{2V_{2n}-V_{2n-1}}{3} = B_n$. Also $\frac{5V_{2n}-V_{2n-1}}{3} = C_n$.*

Proof. Solving the equations $V_{2n-1} = 2C_n - 5B_n$ and $V_{2n} = C_n - B_n$ for B_n and C_n, we get $\frac{(2V_{2n}-V_{2n-1})}{3} = B_n$ and $\frac{(5V_{2n}-V_{2n-1})}{3} = C_n$. Since $U_n = 3B_n$, $2V_{2n} - V_{2n-1} = U_n$ follows. ■

Given an almost balancing number x, an important problem is to find the next one. This job is easy for A_1-balancing numbers since these numbers are proportional to balancing numbers. However, there are two classes of A_2-balancing numbers and hence, we have to find functions transforming an odd order A_2-balancing number to the next even order A_2-balancing number and also an even order A_2-balancing number to the next odd order A_2-balancing number. These are accomplished in the following theorem.

3.5.2 Theorem. *If x is an odd order A_2-balancing number then the next (even order) A_2-balancing number is $\frac{11x+3\sqrt{8x^2-7}}{7}$; further if x is an even order A_2-balancing number then the next (odd order) A_2-balancing number is $\frac{9x+2\sqrt{8x^2-7}}{7}$.*

Proof. Let us first assume that x is an odd order A_2-balancing number. This means that $x = 2C_n - 5B_n$ for some n. Thus, $4C_n^2 = (x + 5B_n)^2$ and using $C_n^2 = 8B_n^2 + 1$, the last equation is transformed into the quadratic equation (in B_n)

$$7B_n^2 - 10B_n x - (x^2 - 4) = 0$$

whose solutions are

$$B_n = \frac{5x + 2\sqrt{8x^2 - 7}}{7}, \quad \frac{5x - 2\sqrt{8x^2 - 7}}{7}.$$

The second solution is not feasible since it is negative for $x > 2$ and is not a positive integer for $x = 1$. Hence the only possibility left is

$$B_n = \frac{5x + 2\sqrt{8x^2 - 7}}{7}.$$

In a similar manner, one can also verify that

$$C_n = \frac{16x + 5\sqrt{8x^2 - 7}}{7}.$$

Thus,

$$C_n - B_n = \frac{11x + 3\sqrt{8x^2 - 7}}{7}$$

which is nothing but the A_2-balancing number next to $x = 2C_n - 5B_n$. This proves the first part. The proof of the second part is similar and hence it is omitted. ■

Given an almost balancing number x, we next consider the problem of finding the previous one. This work is very easy for A_1-balancing numbers. For the two classes of A_2-balancing numbers, we have to find out functions transforming an odd order A_2-balancing number to the previous even order A_2-balancing number and also for an even order A_2-balancing number to the previous odd order A_2-balancing number. In this connection we have the following theorem.

3.5.3 Theorem. *If x is an odd order A_2-balancing number then the previous (even order) A_2-balancing number is $\frac{11x-3\sqrt{8x^2-7}}{7}$; further if x is an even order A_2-balancing number then the previous (odd order) A_2-balancing number is $\frac{9x-2\sqrt{8x^2-7}}{7}$.*

The proof of Theorem 3.5.3 is similar to that of Theorem 3.5.2 and hence it is omitted.

3.6 Some interesting links to balancing and related numbers

In this section, by virtue of modular arithmetic, we develop certain transformations of A_2-balancing numbers resulting in balancing, Lucas-balancing and Pell numbers. If x is an A_2-balancing number then $8x^2-7$ is a perfect square. Our first observation is that

$$x^2 \equiv 8x^2 - 7 \pmod 7.$$

This gives

$$\sqrt{8x^2-7} \equiv \pm x \pmod 7)$$

from which, we infer that for a given A_2-balancing number x, either $\frac{\sqrt{8x^2-7}+x}{7}$ or $\frac{\sqrt{8x^2-7}-x}{7}$ is a natural number. Since

$$8\left[\frac{\sqrt{8x^2-7}\pm x}{7}\right]^2 + 1 = \left[\frac{8x \pm \sqrt{8x^2-7}}{7}\right]^2,$$

either $\frac{\sqrt{8x^2-7}+x}{7}$ or $\frac{\sqrt{8x^2-7}-x}{7}$ is a balancing number. This proves

3.6.1 Theorem. *If x is an A_2-balancing number then either $\frac{\sqrt{8x^2-7}+x}{7}$ or $\frac{\sqrt{8x^2-7}-x}{7}$ is a balancing number.*

Letting $B = \frac{\sqrt{8x^2-7}\pm x}{7}$, we arrive at the quadratic equation

$$7x^2 \pm 2Bx - (7B^2+1) = 0$$

whose solutions are

$$x = \pm B \pm \sqrt{8B^2+1} = \pm B \pm C,$$

where C is the Lucas-balancing number corresponding to B. Thus, the possible values of x are $\pm B_n \pm C_n,\ n = 1,2,\dots$. But, by definition x is positive; hence x has only two choices, namely, $x = C_n + B_n$ or $x = C_n - B_n,\ n = 1,2,\dots$. The A_2-balancing number 1 can be interpreted as $V_0 = C_0 - B_0$. Since, by virtue of results of Section 3.4,

$$\begin{aligned} V_{2n+1} = 2C_{n+1} - 5B_{n+1} &= 2(C_{n+1} - 3B_{n+1}) + B_{n+1} \\ &= -2B_n + B_{n+1} = C_n + B_n, \end{aligned}$$

it follows that all A_2-balancing numbers are of the form $C_n \pm B_n$ and we may alternatively list A_2-balancing numbers as $V_{2n} = C_n - B_n$ and $V_{2n+1} = C_n + B_n$; $n = 1,2, \dots$.

On the other hand, for any A_2-balancing number x,

$$36x^2 \equiv 8x^2 - 7 \text{ (mod 7)}.$$

This gives

$$6x \equiv \pm\sqrt{8x^2 - 7} \text{ (mod 7)}.$$

Thus, either $\frac{6x+\sqrt{8x^2-7}}{7}$ or $\frac{6x-\sqrt{8x^2-7}}{7}$ is a natural number. Since

$$2\left[\frac{6x \pm \sqrt{8x^2-7}}{7}\right]^2 - 1 = \left[\frac{3\sqrt{8x^2-7} \pm 4x}{7}\right]^2,$$

$\frac{6x+\sqrt{8x^2-7}}{7}$ or $\frac{6x-\sqrt{8x^2-7}}{7}$ is an odd Pell number say $P = \frac{6x\pm\sqrt{8x^2-7}}{7}$, which leads to the quadratic equation

$$4x^2 - 12Px + (7P^2 + 1) = 0.$$

This gives $x = \frac{3P\pm\sqrt{2P^2-1}}{2}$ and we have expressed a class of A_2-balancing numbers in terms of odd Pell numbers. Writing $P = P_{2n-1}$ and observing that $\sqrt{2P_{2n-1}^2 - 1} = Q_{2n-1}$, where Q_n is the n^{th} associated Pell number, we arrive at the conclusion that the A_2-balancing numbers are of the form $\frac{3P_{2n-1}\pm Q_{2n-1}}{2}$. Since $\frac{3P_1-Q_1}{2} = 1$ and $\frac{3P_1+Q_1}{2} = 2$, we can easily write the A_2-balancing numbers in terms of odd order Pell and associated Pell number as

$$V_{2n-1} = \frac{3P_{2n-1} + Q_{2n-1}}{2},\ V_{2n} = \frac{3P_{2n+1} - Q_{2n+1}}{2};\ n = 1,2, \dots .$$

3.7 Open problems

Panda [34] introduced the concept of k^{th} order balancing numbers as generalization of balancing numbers. According to him, a natural number n is called a k^{th} order balancing number if $1^k + 2^k + \cdots + (n-1)^k = (n+1)^k + \cdots + (n+r)^k$ holds for some natural number r. He conjectured in [34] that there doesn't exist any such number for $k > 1$. Using the terminology of this chapter, we can call a natural number n, an almost k^{th} order balancing number if $|1^k + 2^k + \cdots + (n-1)^k - (n+1)^k + \cdots + (n+r)^k| = 1$ holds for some r. However, the exploration of almost k^{th} order balancing numbers for $k > 1$ is a more challenging job, the existence of any such number is often doubtful. We leave the study of these numbers to the future researchers.

Chapter 4

Almost Cobalancing Numbers

4.1 Introduction

It is well-known that cobalancing numbers b and cobalancers r are solutions of the Diophantine equation $1 + 2 + \cdots + b = (b + 1) + (b + 2) + \cdots + (b + r)$ [32]. Alternatively, a natural number n is a cobalancing number if and only if $8b^2 + 8b + 1$ is a perfect square or equivalently, the pronic number $b^2 + b$ is triangular. If b is a cobalancing number then $\sqrt{8b^2 + 8b + 1}$ is called a Lucas-cobalancing number. Cobalancing numbers and cobalancers are very closely associated with balancing numbers and balancers [41].

The triangular numbers are very mysterious. Some triangular numbers are squares e.g. $1, 36, 1225$ etc. and their square roots are balancing numbers [3]. Some triangular numbers are pronic e.g. $6, 210, 7140$ etc. and integral parts of their positive square roots are cobalancing numbers. There are also triangular numbers differing from perfect squares by 1 e.g. $10, 15, 120$ etc. and are known as almost square triangular numbers [47]. The approximate square roots of these numbers give rise to an interesting number sequence known as almost balancing numbers already discussed in Chapter 3. One can notice that there are also triangular numbers expressible in the form $n(n + 1) \pm 1$ e.g. $3, 21, 55$ etc.; we can call them *almost pronic triangular numbers*. The approximate square roots of this class of numbers will play an important role in defining a new class of numbers which we call *almost cobalancing numbers*. The whole objective of this chapter is to introduce almost cobalancing numbers and to give a complete look into this class of numbers.

4.2 Definitions and preliminaries

By slightly altering the definition of cobalancing numbers, we define almost cobalancing numbers as follows.

4.2.1 Definition. We call a natural number n an *almost cobalancing number* if

$$|\{(n+1)\ +\ (n+2)\ +\cdots+\ (n+r)\} - \{1+2+\ \cdots\ +n\}\,| = 1$$

for some natural number r, which we call the *almost cobalancer* corresponding to n.

In view of the above definition, we can classify cobalancing numbers in two classes. If

$$\{(n+1)\ +\ (n+2)\ +\cdots+\ (n+r)\} - \{1+2+\ \ldots\ +n\}\ = 1,$$

for some pair of natural numbers (n, r), we call n, an *almost cobalancing number of first kind*; further, if

$$\{(n+1)\ +\ (n+2)\ +\cdots+\ (n+r)\} - \{1+2+\ \ldots\ +n\}\ = -1,$$

we call n, an *almost cobalancing number of second kind*. In the former case, we call r, an *almost cobalancer of first kind,* while in the latter case we call r, an *almost cobalancer of second kind*.

For the sake of simplicity and maintaining similarity with almost balancing numbers, we call almost cobalancing numbers of first kind as A_1-*cobalancing numbers* and almost cobalancers of first kind as A_1-*cobalancers*. Similarly, we call almost cobalancing numbers of second kind as A_2-*cobalancing numbers* and almost cobalancers of second kind as A_2-*cobalancers*.

Observe that if n is an A_1-cobalancing number with A_1-cobalancer r then

$$\frac{n(n+1)}{2} + 1 = \frac{(n+r)(n+r+1)}{2} - \frac{n(n+1)}{2}$$

which reduces to

$$n^2 + n + 1 = \frac{(n+r)(n+r+1)}{2}$$

and the A_1-cobalancer r can be expressed in terms of the A_1-cobalancing number n as

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 8n + 9}}{2}.$$

Thus, a natural number $n > 2$ is an A_1-cobalancing number if and only if $n^2 + n + 1$ is a triangular number or equivalently, $8n^2 + 8n + 9$ is a perfect square.

Similarly, if n is an A_2-cobalancing number with A_2-cobalancer r, then

$$\frac{n(n+1)}{2} - 1 = \frac{(n+r)(n+r+1)}{2} - \frac{n(n+1)}{2}$$

which simplifies to

$$n^2 + n - 1 = \frac{(n+r)(n+r+1)}{2},$$

and the A_2-cobalancer r expressed as a function of the A_2-cobalancing number n is

$$r = \frac{-(2n+1) + \sqrt{8n^2 + 8n - 7}}{2}.$$

Thus, a natural number n is a A_2-cobalancing number if and only if $n^2 + n - 1$ is a triangular number or equivalently, $8n^2 + 8n - 7$ is a perfect square.

4.2.2 Examples. Since $8 \cdot 4^2 + 8 \cdot 4 + 9 = 169$ is a perfect square, 4 is a A_1-cobalancing number with A_1-balancer $r = \frac{-(2.4+1)+\sqrt{8 \cdot 4^2 + 8 \cdot 4 + 9}}{2} = 2$. Similarly, since $8 \cdot 7^2 + 8 \cdot 7 - 7 = 21^2$, 7 is A_2-cobalancing number and it is easy to see that the corresponding A_2-cobalancer is 3. Since $8 \cdot 1^2 + 8 \cdot 1 + 9 = 25$ and $8 \cdot 1^2 + 8 \cdot 1 - 7 = 9$ are perfect squares, we accept 1 as the first A_1- as well as the first A_2-cobalancing numbers just like Panda and Ray [32] accepted 0 as the first cobalancing number.

4.3 Computation of A_1-cobalancing numbers

In Section 4.2, we noticed that $x > 2$ is a A_1-cobalancing number if and only if $8x^2 + 8x + 9$ is a perfect square. To find the Binet or closed form of these numbers, we consider solving the quadratic Diophantine equation $8x^2 + 8x + 9 = Y^2$ which is equivalent to the generalized Pell's equation

$$Y^2 - 2X^2 = 7 \tag{4.1}$$

with $X = 2x + 1$. This equation admits two classes of solutions corresponding to the fundamental solutions $3 + \sqrt{2}$ and $5 + 3\sqrt{2}$. However, we employ the theory of gap-balancing numbers [38] to solve equation (4.1). Observe that the generalized Pell's equation $Y^2 - 2X^2 = 7$ indicates that $2X^2 + 7$ is a perfect square and hence X is a 2-gap balancing number ; so, the possible values of X are $6B_n \pm C_n$ [38]. Thus, the A_1-cobalancing numbers constitute two classes. We denote the n^{th} member of the first class by U_n and that of the second class by V_n. The solutions of (4.1) are thus,

$$U_n = \frac{6B_n - C_n - 1}{2};\ n = 1,2,\ldots \tag{4.2}$$

and

$$V_n = \frac{6B_n + C_n - 1}{2};\ n = 1,2,\ldots\,. \tag{4.3}$$

Observe that $U_1 = \frac{6B_1 - C_1 - 1}{2} = 1$ does not satisfy the defining equation for A_1-cobalancing number, but we accept it as a A_1-cobalancing number just for the sake of developing recurrence relation. Using the shift formulas

$$B_{n\pm1} = 3B_n \pm C_n$$

[see 41, p.25], U_n and V_n can be alternatively written only in terms of balancing numbers as

$$U_n = \frac{3B_n + B_{n-1} - 1}{2};\ n = 1,2,\ldots \tag{4.4}$$

and

$$V_n = \frac{3B_n + B_{n+1} - 1}{2};\ n = 1,2,\ldots\,. \tag{4.5}$$

Using (4.4) and (4.5), we can develop binary recurrences for the sequences $\{U_n\}_{n=1}^{\infty}$ and $\{V_n\}_{n=1}^{\infty}$. In view of (4.4) and the shift formula preceding it, we get

$$\begin{aligned} U_{n+1} + U_{n-1} &= \frac{18B_n + 6B_{n-1} - 6}{2} + 2 \\ &= \frac{3(B_{n+1} + B_{n-1}) + (B_n + B_{n-2}) - 2}{2} \\ &= 6U_n + 2. \end{aligned}$$

Thus, the recurrence relation for the sequence $\{U_n\}$ is given by

$$U_{n+1} = 6U_n - U_{n-1} + 2$$

with initial values $U_0 = -1, U_1 = 1$. ($U_0 = -1$ is obtained by backward calculation.) In a similar manner, one can have.

$$V_{n+1} = 6V_n - V_{n-1} + 2$$

with initial values $V_0 = 0, V_1 = 4$. ($V_0 = 0$ is also obtained by backward calculation.)

The above discussion proves

4.3.1 Theorem. *The values of x satisfying the Diophantine equation* $8x^2 + 8x + 9 = y^2$ *in positive integers partition in two classes. The first class is* given by $U_n = \frac{3B_n + B_{n-1} - 1}{2}$; $n = 1,2, \ldots$ *and the second class is* $V_n = \frac{3B_n + B_{n+1} - 1}{2}$; $n = 1,2, \ldots$. *The sequences* $\{U_n\}$ and $\{V_n\}$ *satisfy the recurrence relations* $U_{n+1} = 6U_n - U_{n-1} + 2$*;* $U_0 = -1, U_1 = 1$ *and* $V_{n+1} = 6V_n - V_{n-1} + 2$*;* $V_0 = 0, V_1 = 4$.

It is interesting to notice that the recurrence relations of the two classes of A_1-cobalancing numbers are identical with that of the cobalancing numbers; the differences are associated only with the initial values.

Henceforth, we call A_1-cobalancing numbers of first class as A_{11}-cobalancing numbers and those in the second class as A_{12}-cobalancing numbers.

The A_1-cobalancing numbers can also be expressed in terms of the cobalancing numbers and Lucas-cobalancing numbers. We first consider the case of A_{11}-cobalancing numbers. These are given by

$$U_n = \frac{6B_n - C_n - 1}{2};\ n = 1,2,\cdots.$$

Since,

$$R_n = \frac{-(2B_n + 1) + C_n}{2}$$

[41, p.18], we have

$$C_n = 2(B_n + R_n) + 1,$$

and using this identity we can rewrite U_n as

$$U_n = 6B_n + 3R_n - 2C_n + 1.$$

Since $B_{n-1} = 3B_n - C_n$ [41, p.25], U_n takes the form

$$U_n = 2B_{n-1} + 3R_n + 1.$$

Taking into account $R_n = b_n$ and $B_{n-1} = r_n$ [41, Theorem 3.6.2], we get

$$U_n = 2r_n + 3b_n + 1.$$

Finally, by virtue of the relationship

$$r_n = \frac{-(2b_n + 1) + c_n}{2}$$

[41, p.34], we can express U_n as $U_n = b_n + c_n, n = 1,2, \dots$. Proceeding in a similar manner, it is easy to see that the A_{12}-cobalancing numbers can be alternatively written as $V_n = c_n - b_n - 1, n = 1,2, \dots$.

4.4 Computation of A_2-cobalancing numbers

Our observation in Section 4.2 is that, if x is a A_2-cobalancing number then $8x^2 + 8x - 7$ is a perfect square and once we accept 1 as a A_2-cobalancing number, we can assert that a positive integer x is an A_2-cobalancing numbers if and only $8x^2 + 8x - 7$ is a perfect square. Thus, to find all A_2-cobalancing numbers, we need to solve the quadratic Diophantine equation $8x^2 + 8x - 7 = y^2$ in positive integers. This equation reduces to the generalized Pell's equation $y^2 - 2(2x + 1)^2 = -9$. However, to solve this equation, we apply the theory of cobalancing numbers which facilitates expressing the A_2-cobalancing numbers in terms of the cobalancing numbers.

The Diophantine equation $8x^2 + 8x - 7 = y^2$ can be written in the form

$$y^2 + 9 = 2(2x + 1)^2. \tag{4.6}$$

We claim that $2x + 1 \equiv 0 (\text{mod } 3)$. Observe that if $2x + 1 \equiv \pm 1 (\text{mod } 3)$, then $y^2 \equiv 2 (\text{mod } 3)$ which is impossible since 2 is not a quadratic residue modulo 3. Thus, $2x + 1 \equiv 0 (\text{mod } 3)$ and consequently, $y \equiv 0 (\text{mod } 3)$. Writing $2x + 1 = 3k$ and $y = 3l$, equation (4.6) reduces to

$$l^2 + 1 = 2k^2. \tag{4.7}$$

Observing that l is odd and using simple algebra, equation (4.7) can be finally written in terms of a Pythagorean equation as

$$w^2 + (w + 1)^2 = k^2$$

with $w = (l - 1)/2$. Thus, k is odd and

$$\frac{w(w + 1)}{2} = \frac{k - 1}{2} \cdot \frac{k + 1}{2}.$$

Since $\frac{k-1}{2}$ and $\frac{k+1}{2}$ are consecutive numbers and $\frac{w(w+1)}{2}$ is a triangular number, $(k - 1)/2$ must be a cobalancing number [32, p.1190]. Thus, $k = 2b + 1$ and since $2x + 1 = 3k$, it follows that $x = 3b + 1$. Denoting the n^{th} A_2-cobalancing number by W_n, we have $W_n = 3b_n + 1, n = 1,2, \ldots$.

We summarize the above discussion in the following theorem.

4.4.1 Theorem. *The values of x satisfying the Diophantine equation $8x^2 + 8x - 7 = y^2$ in positive integers are given by $W_n = 3b_n + 1, n = 1,2, \ldots$, where b_n is the n^{th} cobalancing number. Thus, the A_2-cobalancing numbers are given by $3b_n + 1, n = 1,2, \ldots$.*

Using the recurrence relation for cobalancing numbers, it is easy to see that the n^{th} A_2-cobalancing numbers satisfy a recurrence relation identical with cobalancing numbers. In particular,

$$W_{n+1} = 6W_n - W_{n-1} + 2$$

with initial values $W_0 = W_1 = 1$.

4.5 Transformations from almost cobalancing numbers to cobalancing and balancing numbers

In the last two sections, we explored all the almost cobalancing numbers. In the present section, we consider finding functions mapping almost cobalancing numbers to cobalancing and balancing numbers.

The mappings from cobalancing numbers to A_2-cobalancing numbers and A_2-cobalancing numbers to cobalancing numbers are obvious. For example, if x is a cobalancing number then $3x + 1$ is a A_2-cobalancing number and if x is a A_2-cobalancing number then $(x - 1)/3$ is a cobalancing number. Again, if x is a cobalancing number then the next one is $3x + \sqrt{8x^2 + 8x + 1}$ +1 [6, p.37]. Thus, if y is a A_2-cobalancing number then the next one is $3y + \sqrt{8y^2 + 8y - 7} + 1$ since, in this case, $y = 3x + 1$, where x is a cobalancing number.

We next identify functions that transform A_1-cobalancing numbers to cobalancing numbers. In the last section, we have already proved that the A_{11}-cobalancing numbers are of the form $b_n + c_n$, while A_{12}-cobalancing numbers are of the form $c_n - b_n - 1$. By virtue of the relation $c_n = \sqrt{8b_n^2 + 8b_n + 1}$, the two classes of A_1-cobalancing numbers are expressible in the forms

$$U_n = b_n + \sqrt{8b_n^2 + 8b_n + 1}$$

and

$$V_n = -(b_n + 1) + \sqrt{8b_n^2 + 8b_n + 1}$$

respectively.

The above discussion results in the following theorem.

4.5.1 Theorem. *If x is a cobalancing number then $\alpha(x) = x + \sqrt{8x^2 + 8x + 1}$ is a A_{11}-cobalancing number while $\beta(x) = -(x+1) + \sqrt{8x^2 + 8x + 1}$ is a A_{12}-cobalancing number.*

Now we come to our main problem of finding functions that transform A_1-cobalancing numbers to cobalancing numbers. It is easy to check that the functions $\alpha(x)$ and $\beta(x)$ are strictly increasing in the domain $[1, \infty)$. Hence their inverses exist and are equal to

$$\alpha^{-1}(y) = \frac{-(y+4) + \sqrt{8y^2 + 8y + 9}}{7}$$

and

$$\beta^{-1}(y) = \frac{(y-3) + \sqrt{8y^2 + 8y + 9}}{7}.$$

The above discussion leads to the following theorem.

4.5.2 Theorem. *If x is an A_{11}-cobalancing number then $f(x) = \frac{-(x+4)+\sqrt{8x^2+8x+9}}{7}$ is a cobalancing number, while if x is an A_{12}-cobalancing number then $g(x) = \frac{(x-3)+\sqrt{8x^2+8x+9}}{7}$ is a cobalancing number.*

Using the expressions for A_{11}- and A_{12}-cobalancing numbers given in (4.3) and (4.4), it is easy to see that if x is a balancing number then

$$\gamma(x) = \frac{6x - \sqrt{8x^2 + 1} - 1}{2}$$

is an A_{11}-cobalancing number and

$$\delta(x) = \frac{6x + \sqrt{8x^2 + 1} - 1}{2}$$

is an A_{12}-cobalancing number. The inverses of these functions exist and are equal to

$$\gamma^{-1}(x) = \frac{3(2x+1)+\sqrt{8x^2+8x+9}}{14},\ \delta^{-1}(x) = \frac{3(2x+1)-\sqrt{8x^2+8x+9}}{14}.$$

The above discussion proves the following theorem.

4.5.3 Theorem. *If x is an A_{11}-cobalancing number then $h(x) = \frac{3(2x+1)+\sqrt{8x^2+8x+9}}{14}$ is a balancing number, while if x is an A_{12}-cobalancing number then $t(x) = \frac{3(2x+1)-\sqrt{8x^2+8x+9}}{14}$ is a balancing number.*

Finally, we establish functions mapping A_{11}-cobalancing numbers to A_{12}-cobalancing numbers and vice versa. These functions are contained in the following theorem.

4.5.4 Theorem. *If x is an A_{11}-cobalancing number then $\tau(x) = \frac{(9x+1)-2\sqrt{8x^2+8x+9}}{7}$ is an A_{12}-cobalancing number and if x is an A_{12}-cobalancing number then $\varphi(x) = \frac{(9x+1)+2\sqrt{8x^2+8x+9}}{7}$ is an A_{11}-cobalancing number.*

Proof. First assume that x is an A_{11}-cobalancing number. This means that $x = b_n + c_n$ for some n. Thus, $c_n^2 = (x - b_n)^2$ and using $c_n^2 = 8b_n^2 + 8b_n + 1$, the last equation can be transformed into the quadratic equation (in b_n) as

$$7b_n^2 + (2x+8)b_n + (1-x^2) = 0$$

whose solutions are

$$b_n = \frac{-(x+4)+\sqrt{8x^2+8x+9}}{7}, \frac{-(x+4)-\sqrt{8x^2+8x+9}}{7}.$$

The second solution is not feasible since it is negative for $x > 1$ and is not a positive integer for $x = 1$. Hence,

$$b_n = \frac{-(x+4)+\sqrt{8x^2+8x+9}}{7}.$$

Similarly, one can check that

$$c_n = \frac{(8x+4) - \sqrt{8x^2+8x+9}}{7}.$$

Finally,

$$c_n - b_n - 1 = \frac{(9x+1) - 2\sqrt{8x^2+8x+9}}{7} = \tau(x)$$

proves that $\tau(x)$ is an A_{12}-cobalancing number for any given A_{11}-cobalancing number x. Applying a similar technique, one can also prove that for any A_{12}-cobalancing number x, $\varphi(x)$ is an A_{11}-cobalancing number. ■

4.6 Application to a Diophantine equation

In this section, we establish the association of solutions of a Pythagorean equation with A_2-cobalancing numbers. We start with the Pythagorean equation

$$x^2 + (x+3)^2 = y^2$$

which we can rewrite in the form

$$(2x+3)^2 + 9 = 2y^2. \tag{4.8}$$

Since $(2x+3)^2 \equiv 1 \pmod 4$ and $9 \equiv 1 \pmod 4$, it follows that $y^2 \equiv 1 \pmod 4$. Hence y is odd, say $y = 2k+1$ where k is a positive integer. Substitution in (4.8) gives

$$8k^2 + 8k - 7 = (2x+3)^2. \tag{4.9}$$

Thus, k is an A_2-cobalancing number say $k = W$. This gives $y = 2W + 1$. Substituting in (4.8), we get

$$x = \frac{\sqrt{8W^2+8W-7} - 3}{2}.$$

The above discussion proves

4.6.1 Theorem. *The solutions of the Pythagorean equation* $x^2 + (x+3)^2 = y^2$ *can be written in terms of the* A_2*-cobalancing numbers as* $x = \frac{\sqrt{8W_n^2+8W_n-7}-3}{2}$, $y = 2W_n + 1$, $n = 1,2, \ldots$ *where* W_n *is the* n^{th} A_2*-cobalancing number.*

4.7 Open problems

Panda [34] also introduced the concept of k^{th} order cobalancing numbers as generalization of cobalancing numbers. He calls a natural number n, a k^{th} order cobalancing number if $1^k + 2^k + \cdots + n^k = (n+1)^k + \cdots + (n+r)^k$ holds for some r. He also conjectured that there doesn't exist any such number for $k > 1$. Analogues to the definition of almost k^{th} order balancing number given in Section 3.7, it is possible to define k^{th} order almost cobalancing numbers. However, the exploration of k^{th} order almost cobalancing numbers for $k > 1$ is also a more challenging job and the existence of any such number is doubtful. We keep the study of k^{th} order almost cobalancing numbers as an open problem for the future researchers.

Chapter 5

Almost Balancing-like Sequences

5.1 Introduction

It is well known that the pair (n, r), where n is a balancing number and r is the corresponding balancer, is a solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + \cdots + (n+r). \tag{5.1}$$

In Chapter 3, we introduced the concept of almost balancing numbers as generalizations of balancing numbers. We defined the almost balancing numbers n and corresponding almost balancers r are as solutions of a Diophantine equation which can be formed by setting the difference of left hand and right hand sides of (5.1) equal to 1.

The sequence of balancing numbers $\{B_n\}_{n=1}^{\infty}$ is also solutions of the binary recurrence $B_{n+1} = 6B_n - B_{n-1}$ with initial values $B_0 = 0$ and $B_1 = 1$. As generalizations of the balancing sequence, Panda and Rout [37] studied a class of binary recurrences defined by $x_{n+1} = Ax_n - x_{n-1}$, $x_0 = 0, x_1 = 1$, where $A > 2$ is any natural number. They showed that these sequences enjoy many properties identical with those of the balancing sequence and these sequences are subsequently known as balancing-like sequences. Since for $A = 2$ (though this case is excluded from the definition balancing-like sequences), the positive solutions of the binary recurrence $x_{n+1} = Ax_n - x_{n-1}$, $x_0 = 0, x_1 = 1$ are the natural numbers, the balancing-like sequences are thought of as generalizations of the sequence of natural numbers and are sometimes termed as natural sequences. Khan and Kwong [20] called these sequences as *generalized natural number sequences*.

Panda and Rout [37] proved that for a fixed natural number A, x is a balancing-like number if and only if Dx^2+1, where $D=\frac{A^2-4}{4}$, is a perfect rational square and $y=\sqrt{Dx^2+1}$ is known as a Lucas-balancing-like number. Lucas-balancing-like numbers have integral values if A is even.

In this chapter, as variants of the balancing-like sequences, we introduce the concept of almost balancing-like sequences. These generalizations are at par with the generalization of balancing sequence to the balancing-like sequences. It is well known that a natural number x is a balancing number if and only if $8x^2+1$ is a perfect square [3]. While generalizing the balancing sequence to the almost balancing sequence, we altered the basic defining equation (5.1) and after simplification we observed that a natural number x is an almost balancing number if and only if $8(x^2\pm 1)+1$ is a perfect square. This observation helps us in defining almost balancing-like sequences since balancing-like numbers do not have a defining equation like (5.1).

5.2 Almost balancing-like numbers

Let $A>2$ be a fixed natural number. We call a natural number x an *almost balancing-like number* if and only if $D(x^2\pm 1)+1$ is a perfect rational square. Similar to the case of almost balancing numbers, we call x an A_1-*almost balancing-like number* if $D(x^2+1)+1$ is a perfect rational square while if $D(x^2-1)+1$ is a perfect rational square, we call x an A_2-*almost balancing-like number*.

Let $A>2$ be even so that D is a natural number. If x is an A_1-almost balancing-like number then Dx^2+D+1 is a perfect square say $Dx^2+D+1=y^2$ and this equation can be rewritten as

$$y^2-Dx^2=D+1=\frac{A^2}{4} \tag{5.2}$$

which is a generalized Pell's equation. Similarly, if x is an A_2-almost balancing-like number then $Dx^2-D+1=y^2$ can be rewritten as the generalized Pell's equation

$$y^2-Dx^2=-D+1. \tag{5.3}$$

Now let $A > 2$ be odd so that $D = \frac{A^2-4}{4}$ is not a natural number. If x is an A_1-almost balancing-like number then $4(Dx^2 + D + 1)$ is a perfect integral square say, $4(Dx^2 + D + 1) = y^2$ which can be written in terms of a generalized Pell's equation as

$$y^2 - (A^2 - 4)x^2 = A^2. \tag{5.4}$$

Likewise, If x is an A_2-almost balancing-like number then $4(Dx^2 - D + 1)$ is a perfect integral square say $4(Dx^2 - D + 1) = y^2$ and this equation can be transformed into the generalized Pell's equation

$$y^2 - (A^2 - 4)x^2 = -A^2 + 8. \tag{5.5}$$

Thus, for each $A > 2$, exploring two types of almost balancing-like numbers reduces to solving two generalized Pell's equations. In the subsequent sections, we will explore almost balancing-like sequences corresponding to certain values of A.

5.3 Almost balancing-like sequence: $A = 3$

It is well-known that the Fibonacci and Lucas sequences are defined by means of the binary recurrences

$$F_{n+1} = F_n + F_{n-1},\ L_{n+1} = L_n + L_{n-1}$$

respectively with initial values $F_0 = 0,\ F_1 = 1, L_0 = 2,\ L_1 = 1$. The Binet forms of these sequences are

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},\ L_n = \alpha^n + \beta^n$$

where $\alpha = \frac{1+\sqrt{5}}{2}$, which is commonly known as the golden ratio [29] and $\beta = \frac{1-\sqrt{5}}{2}$.

The balancing-like numbers corresponding to $A = 3$ are solutions of the binary recurrence

$$x_{n+1} = 3x_n - x_{n-1}$$

with initial values $x_0 = 0, x_1 = 1$ and have the Binet form

$$x_n = \frac{k^n - l^n}{\sqrt{5}}, n = 1,2, \ldots$$

where $k = \frac{3+\sqrt{5}}{2}$ and $l = \frac{3-\sqrt{5}}{2}$. On careful examination, one can realize that $k = \alpha^2$ and $l = \beta^2$. Hence, the Binet form for balancing-like sequence corresponding to $A = 3$ takes the form

$$x_n = \frac{\alpha^{2n} - \beta^{2n}}{\sqrt{5}}, n = 1,2, \dots .$$

Consequently, $x_n = F_{2n}$, that is, the n^{th} term of this balancing-like sequence is nothing but the $2n^{\text{th}}$ term of the Fibonacci sequence.

To explore A_1- and A_2-almost balancing-like numbers corresponding to $A = 3$, we need to solve the generalized Pell's equations (particular cases of (5.4) and (5.5))

$$y^2 - 5x^2 = 9 \tag{5.6}$$

and

$$y^2 - 5x^2 = -1 \tag{5.7}$$

respectively. Since, we denote the n^{th} balancing-like number by x_n, we prefer to denote the n^{th} term of the A_1-almost balancing-like subsequences arising out of different fundamental solutions of (5.6) by u_n, u_n', u_n'', $\cdots$ while the corresponding values of y by v_n, v_n', v_n'', $\cdots$. Similarly, we also denote the n^{th} term of the A_2-almost balancing-like subsequences arising out of different fundamental solutions of (5.7) by r_n, r_n', r_n'', ... while the corresponding values of y by s_n, s_n', s_n'', ... respectively.

Equations (5.6) and (5.7) have the fundamental solutions $27 + 12\sqrt{5}$ and $2 + \sqrt{5}$ respectively and the fundamental solution of

$$y^2 - 5x^2 = 1 \tag{5.8}$$

is $9 + 4\sqrt{5}$. Hence, the general solution of (5.6) is

$$v_n + \sqrt{5}u_n = \left(27 + 12\sqrt{5}\right)\left(9 + 4\sqrt{5}\right)^{n-1}, n = 1,2, \dots$$

while the general solution of (5.7) is

$$s_n + \sqrt{5}r_n = \left(2 + \sqrt{5}\right)\left(9 + 4\sqrt{5}\right)^{n-1}, n = 1,2, \dots .$$

Thus, the Binet forms for A_1-almost balancing-like numbers corresponding to $A = 3$ is

$$u_n = \frac{(27 + 12\sqrt{5})(9 + 4\sqrt{5})^{n-1} - (27 - 12\sqrt{5})(9 - 4\sqrt{5})^{n-1}}{2\sqrt{5}} \tag{5.9}$$

while the same for A_2-almost balancing-like numbers is

$$r_n = \frac{(2 + \sqrt{5})(9 + 4\sqrt{5})^{n-1} - (2 - \sqrt{5})(9 - 4\sqrt{5})^{n-1}}{2\sqrt{5}} \qquad 5.10)$$

$n = 1,2, \dots$. Using the Binet form of Fibonacci numbers, (5.9) and (5.10) can be simplified as

$$u_n = 3 \cdot \frac{\alpha^{6n} - \beta^{6n}}{2\sqrt{5}} = \frac{3F_{6n}}{2} \tag{5.11}$$

and

$$r_n = \frac{\alpha^{6n-3} - \beta^{6n-3}}{2\sqrt{5}} = \frac{F_{6n-3}}{2} \tag{5.12}$$

$n = 1,2, \dots$. Taking into account $x_n = F_{2n}$, we can express the A_1-almost balancing-like numbers corresponding to $A = 3$ given in (5.11) as

$$u_n = \frac{3x_{3n}}{2}, n = 1,2, \dots \tag{5.13}$$

and using the identity $F_n = F_{n+1} - F_{n-1}$, we can write the A_2-almost balancing-like numbers as

$$r_n = \frac{x_{3n-1} - x_{3n-2}}{2}, n = 1,2, \dots . \tag{5.14}$$

Using (5.13), (5.14) and the binary recurrence of the balancing-like sequence corresponding to $A = 3$, it can be shown that the A_1- and A_2-almost balancing-like sequences are solutions of the binary recurrences

$$r_{n+1} = 18r_n - r_{n-1}, s_{n+1} = 18s_n - s_{n-1}$$

respectively with initial values $r_0 = 0, r_1 = 12, s_0 = 1, s_1 = 17$.

The above discussion regarding almost balancing-like numbers corresponding to $A = 3$ can be summarized as follows.

5.3.1 Theorem. *The A_1- and A_2-almost balancing-like numbers corresponding to $A = 3$ are solutions in x of the generalized Pell's equations $y^2 - 5x^2 = 9$ and $y^2 - 5x^2 = -1$ respectively. These numbers obey the binary recurrences $r_{n+1} = 18r_n - r_{n-1}$, $s_{n+1} = 18s_n - s_{n-1}$ respectively with initial terms $r_0 = 0, r_1 = 12, s_0 = 1, s_1 = 17$. The n^{th} A_1-almost balancing-like numbers are given by $u_n = \frac{3F_{6n}}{2}$ while the n^{th} A_2-almost balancing-like numbers are of the form $r_n = \frac{F_{6n-3}}{2}$, $n = 1,2, \dots$.*

Expressed as numerical sequences, the balancing-like sequence corresponding to $A = 3$ is

$$1, 3, 8, 21, 55, 144, \dots,$$

the A_1- almost balancing-like sequence is

$$12, 216, 3876, 69552, 1248060, \dots,$$

and finally the A_2- almost balancing-like sequence is

$$1, 17, 305, 5473, 98209, 1762289, \dots .$$

5.4 Almost balancing-like sequence: $A = 4$

The balancing-like numbers corresponding to $A = 4$ are solutions of the binary recurrence

$$x_{n+1} = 4x_n - x_{n-1}$$

with initial values $x_0 = 0, x_1 = 1$ and have the Binet form

$$x_n = \frac{k^n - l^n}{\sqrt{5}}, n = 0,1,2, \dots \tag{5.15}$$

where $k = 2 + \sqrt{3}$ and $l = 2 - \sqrt{3}$.

To find out the A_1- and A_2-almost balancing-like numbers corresponding to $A = 4$, we need to solve the generalized Pell's equations

$$y^2 - 3x^2 = 4 \tag{5.16}$$

and

$$y^2 - 3x^2 = -2 \tag{5.17}$$

respectively. The generalized Pell's equations (5.16) and (5.17) have fundamental solutions $4 + 2\sqrt{3}$ and $1 + \sqrt{3}$ respectively, while the Pell's equation

$$y^2 - 3x^2 = 1$$

has the fundamental solution $2 + \sqrt{3}$. Using the notations used in the last section, the general solutions of (5.15) and (5.16) can be found from

$$v_n + \sqrt{3}u_n = \left(4 + 2\sqrt{3}\right)\left(2 + \sqrt{3}\right)^{n-1} = 2\left(2 + \sqrt{3}\right)^n$$

$$s_n + \sqrt{3}r_n = \left(1 + \sqrt{3}\right)\left(2 + \sqrt{3}\right)^{n-1} = \left(2 + \sqrt{3}\right)^n - \left(2 + \sqrt{3}\right)^{n-1}$$

respectively. Now, taking into account the Binet form (5.14), the A_1- almost balancing-like numbers corresponding to $A = 4$ are

$$u_n = 2x_n, n = 1,2, \ldots$$

and the A_2-almost balancing-like numbers are

$$r_n = x_n - x_{n-1}, n = 1,2, \ldots .$$

It is easy to check that the A_1- and A_2-almost balancing-like numbers corresponding to $A = 4$ have the same binary recurrences as that of the corresponding balancing-like sequence, however, with different initial values.

The above discussion leads to the following theorem.

5.4.1 Theorem. *The A_1- and A_2-almost balancing-like numbers corresponding to $A = 4$ can be realized as solutions in x of the generalized Pell's equations $y^2 - 3x^2 = 4$ and $y^2 - 3x^2 = -2$ respectively. These two classes of numbers are also solutions of the binary recurrences $r_{n+1} = 4r_n - r_{n-1}$, $s_{n+1} = 4s_n - s_{n-1}$ respectively with initial values $r_0 = 0$, $r_1 = 2, s_0 = 1, s_1 = 1$. The A_1-almost balancing-like numbers are given by $u_n = 2x_n$ while the A_2-almost balancing-like numbers are $r_n = x_n - x_{n-1}$, $n = 1,2, \ldots$.*

5.5 Almost balancing-like sequence: $A = 5$

By definition, the almost balancing-like numbers associated with $A = 5$ are solutions of the binary recurrence

$$x_{n+1} = 5x_n - x_{n-1}$$

with initial values $x_0 = 0, x_1 = 1$ and have the Binet form

$$x_n = \frac{k^n - l^n}{\sqrt{21}}, n = 0,1,2, \ldots \tag{5.18}$$

where $k = \frac{5+\sqrt{21}}{2}$ and $l = \frac{5-\sqrt{21}}{2}$. In view of (5.4) and (5.5), the A_1- and A_2-almost balancing-like numbers corresponding to $A = 5$ are solutions of the generalized Pell's equations

$$y^2 - 21x^2 = 25 \tag{5.19}$$

and

$$y^2 - 21x^2 = -17 \tag{5.20}$$

respectively. The fundamental solution of the Pell's equation

$$y^2 - 21x^2 = 1 \tag{5.21}$$

is $55 + 12\sqrt{21}$ while there are three fundamental solutions of (5.19), they are $19 + 4\sqrt{21}$, $37 + 8\sqrt{21}$ and $275 + 60\sqrt{21}$. Using the notations proposed in Section 5.3, three classes of solutions of (5.19) can be obtained from

$$v_n + \sqrt{21}u_n = \left(19 + 4\sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1},$$

$$v_n' + \sqrt{21}u_n' = \left(37 + 8\sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1}$$

$$v_n'' + \sqrt{21}u_n'' = \left(275 + 60\sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1} = 5\left(55 + 12\sqrt{21}\right)^{n}.$$

Thus, the Binet form for the three classes of A_1-almost balancing-like numbers corresponding to $A = 5$ are

$$u_n = \frac{\left(19 + 4\sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1} - \left(19 - 4\sqrt{21}\right)\left(55 - 12\sqrt{21}\right)^{n-1}}{2\sqrt{21}},$$

$$u_n' = \frac{\left(37 + 8\sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1} - \left(37 - 8\sqrt{21}\right)\left(55 - 12\sqrt{21}\right)^{n-1}}{2\sqrt{21}}$$

and

$$u_n'' = 5 \cdot \frac{\left(55 + 12\sqrt{21}\right)^n - \left(55 - 12\sqrt{21}\right)^n}{2\sqrt{21}}.$$

Since

$$55 + 12\sqrt{21} = k^3,\ \ 55 - 12\sqrt{21} = l^3,$$

$$19 + 4\sqrt{21} = 8k - 1,\ 19 - 4\sqrt{21} = 8l - 1,$$

we can write the first class of A_1-almost balancing-like numbers corresponding to $A = 5$ in terms of the corresponding balancing-like numbers as

$$u_n = \frac{8x_{3n+1} - x_{3n}}{2}, n = 0,1,2, \dots .$$

Similarly, the other two classes of A_1-almost balancing-like numbers can be expressed as

$$u_n' = \frac{16x_{3n+1} - 3x_{3n}}{2}, n = 0,1,2, \dots$$

and

$$u_n'' = \frac{5x_{3n}}{2}, n = 1,2, \dots$$

respectively.

The summary of the above discussion is the following theorem.

5.5.1 Theorem. *The A_1-almost balancing-like numbers corresponding to $A = 5$ are solutions in x of the generalized Pell's equation $y^2 - 21x^2 = 25$. These numbers partition in three classes and are given by $u_n = \frac{8x_{3n+1}-x_{3n}}{2}$, $u_n' = \frac{16x_{3n+1}-3x_{3n}}{2}$, $u_n'' = \frac{5x_{3n}}{2}$, $n = 1,2, \dots$ respectively.*

To explore the A_2-almost balancing-like numbers corresponding to $A = 5$, we need to solve the generalized Pell's equation (5.19), which have two fundamental solutions $2 + \sqrt{21}$ and $142 + 31\sqrt{21}$. As usual, using the notations discussed in Section 5.3, we can find two classes of solutions of (5.19) from

$$s_n + \sqrt{21}r_n = \left(2 + \sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1}$$

and

$$s'_n + \sqrt{21} r'_n = \left(142 + 31\sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1}.$$

Hence, the Binet forms for the two classes of A_2-almost balancing-like numbers corresponding to $A = 5$ are

$$r_n = \frac{\left(2 + \sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1} - \left(2 - \sqrt{21}\right)\left(55 - 12\sqrt{21}\right)^{n-1}}{2\sqrt{21}}$$

and

$$r'_n = \frac{\left(142 + 31\sqrt{21}\right)\left(55 + 12\sqrt{21}\right)^{n-1} - \left(142 - 31\sqrt{21}\right)\left(55 - 12\sqrt{21}\right)^{n-1}}{2\sqrt{21}}$$

respectively. Using methods similar to the computation of A_1-almost balancing-like number, one can easily get

$$r_n = \frac{2x_{3n+1} - 3x_{3n}}{2}, n = 0,1,2, \ldots$$

and

$$r'_n = \frac{62x_{3n+1} - 13x_{3n}}{2}, n = 0,1,2, \ldots .$$

The above discussion proves

5.5.2 Theorem. *The A_2-almost balancing-like numbers corresponding to $A = 5$ are solutions in x of the generalized Pell's equations $y^2 - 21x^2 = -17$. These numbers fall in two classes given by $r_n = \frac{2x_{3n+1} - 3x_{3n}}{2}$ and $r'_n = \frac{62x_{3n+1} - 13x_{3n}}{2}, n = 0,1,2, \ldots$ respectively.*

5.6 Almost balancing-like sequence: $A = 6$

The balancing-like sequence corresponding to $A = 6$ is nothing but the sequence of balancing numbers and hence the corresponding almost balancing-like sequence is the sequence of almost balancing numbers. We have already studied these numbers in Chapter 3 in great detail. What we have observed is that the A_1-almost balancing numbers x are solutions of the generalized Pell's equation $y^2 - 8x^2 = 9$, while the A_2-almost balancing numbers are values of x satisfying $y^2 - 8x^2 = -7$. Further, A_1-almost balancing numbers are thrice the corresponding balancing numbers while, the A_2-almost balancing numbers partition into two classes and are of the form $2B_n - B_{n-1}$ and $B_n - 2B_{n-1}, n = 1,2, \dots$.

5.7 Directions for further research

In Section 2, we observed that the almost balancing-like numbers x corresponding to an even A (> 2) are solutions of the generalized Pell's equations (5.2) and (5.3), the former being associated with A_1-almost balancing-like number, while the latter with A_2-almost balancing-like numbers. Since parametric generalized Pell's equations cannot be solved completely in general, in this section, we extract few subclasses of A_1- almost balancing-like numbers.

In Sections 5.4 and 5.6, we have seen that for any given even positive integer A, there is a subclass consisting of A_1-almost balancing-like numbers that are multiples of the corresponding balancing-like numbers. We are going to show that there is such a class associated with every almost balancing-like sequence associated with even integral values $A > 2$.

Let $A > 2$ be even and consider the subclass of solutions of (5.2) corresponding to those x that are multiple of $\frac{A}{2}$. If $x = \frac{Au}{2}$ for some u then y must also be a multiple of $\frac{A}{2}$ say $y = \frac{Av}{2}$. Substitution in (5.2) gives

$$v^2 - Du^2 = 1,$$

which is nothing but the Pell's equation of the balancing-like sequence corresponding to the same value of A. Thus, for any even value of $A > 2$, one class of almost balancing-like numbers is given by $\left\{\frac{Ax_n}{2}\right\}_{n=1}^{\infty}$, where $\{x_n\}_{n=1}^{\infty}$ is the corresponding sequence of balancing-like numbers.

If A is odd then the A_1-almost balancing-like numbers are values of x satisfying the generalized Pell's equation (5.4) which can be rewritten as

$$y^2 - 4Dx^2 = A^2 \tag{5.22}$$

and a there is also a particular subclass of A_1-almost balancing-like numbers that are multiple of A. Setting $x = Au$ and hence $y = Av$, we can reduce (5.22) to the Pell's equation

$$v^2 - D(2u)^2 = 1. \tag{5.23}$$

Thus, $2u$ is an even balancing-like number. Since the even balancing-like numbers of any balancing-like sequence associated with an odd A have indices that are multiple of 3, it follows that $2u = x_{3n}$. Thus, the subclass of A_1-almost balancing-like numbers that are multiple of A is

$$x = Au = \frac{Ax_{3n}}{2}, n = 1,2, \dots .$$

Exploration of any subclasses of A_2-almost balancing-like numbers for an arbitrary A is not so easy, but with further study of some more particular cases, we believe that this may be possible. We leave it as an open problem for the future researchers.

Chapter 6

Circular Balancing Numbers

6.1 Introduction

A balancing number is a natural number n such that if it is removed from first m ($m > n$ and m depends on n) natural numbers arranged in a line, then the sum of numbers to the left of n is equal to the sum to its right. Several generalizations of balancing numbers have been studied by many authors [5, 26, 33, 38, 43]. In Chapters 3, 4 and 5, we discussed three such generalizations. In this chapter, our focus is on another exciting generalization of balancing numbers, which we call *circular balancing numbers*.

Consider an arrangement of m natural numbers equally spaced on a circle. Fix a number k on this circle. If we delete the two numbers corresponding to a chord whose one end is k and other end is x ($> k$) then the circular arrangement of numbers will be divided into two arcs. If the sums of numbers on these two arcs are same, then we call x a *k-circular balancing number*. More precisely, we can define circular balancing numbers as follows.

6.1.1 Definition. Let k be a fixed positive integer. We call a positive integer n, a k-circular balancing number if the Diophantine equation

$$\begin{aligned}&(k+1)+(k+2)+\cdots+(n-1)\\&=(n+1)+(n+2)+\cdots+m+(1+2+\cdots+k-1)\end{aligned} \tag{6.1}$$

holds for some m.

It is possible to simplify equation (6.1) as

$$T_m + k^2 = n^2, k+2 < n < m,$$

where T_m is the m^{th} triangular number. The Diophantine equation $T_m + k^2 = n^2$ is a variant of the Pythagorean equation $x^2 + y^2 = z^2$ with one square replaced by a triangular number. However, unlike the Pythagorean equation, it is difficult to find a compact form of solutions for the equation $T_m + k^2 = n^2$.

Observe that if $k = 0$, then the circular balancing numbers are nothing but balancing numbers [41]. If $k = 1$, then the circular balancing numbers are almost balancing numbers discussed in Chapter 3.

6.1.2 Examples. Since $2 + 3 = 5$, 4 is a 1-circular balancing number. Similarly, since $11 + \cdots + 19 = 21 + \cdots + 24 + (1 + \cdots + 9)$, 20 is a 10-circular balancing number.

6.2 2-Circular balancing numbers

By definition, a natural number x is a 2-circular balancing number if

$$3 + 4 + (x - 1) = (x + 1) + \cdots + m + 1$$

holds for some natural number m. Equivalently, a natural number $x > 2$ is a 2-circular balancing number if and only if $8x^2 - 31$ is a perfect square. Setting $8x^2 - 31 = y^2$, the calculation of 2-circular balancing numbers reduces to solving the generalized Pell's equation

$$y^2 - 8x^2 = -31. \tag{6.2}$$

It is easy to see that the fundamental solution of the Pell's equation $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$ and $1 + 2\sqrt{8}$ is a fundamental solution of (6.2). Using the theory of generalized Pell's equation discussed in Chapter 2, one class of 2-circular balancing numbers can be obtained from

$$y_n + \sqrt{8}x_n = \left(1 + 2\sqrt{8}\right)\left(3 + \sqrt{8}\right)^{n-1}, \qquad n = 1,2, \dots .$$

Thus, the n^{th} member of this class of 2-circular balancing numbers is given by

$$x_n = \frac{\left(1 + 4\sqrt{2}\right)\left(3 + 2\sqrt{2}\right)^{n-1} - \left(1 - 4\sqrt{2}\right)\left(3 - 2\sqrt{2}\right)^{n-1}}{4\sqrt{2}}, n = 1,2, \dots .$$

Using the Binet form for balancing numbers, one can have

$$x_n = 2B_n - 5B_{n-1}, n = 1,2, \dots .$$

It is well-known that $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of the generalized Pell's equation (6.2). Since $x_n = 2B_n - 5B_{n-1}$ and $B_{-n} = -B_n$, it follows that

$$\begin{aligned} x'_n = x_{-n} &= 2B_{-n} - 5B_{-n-1} \\ &= 5B_{n+1} - 2B_n \end{aligned}$$

is positive and greater than 2 for $n = 0, 1, 2, \cdots$ and hence represents another class of 2-circular balancing numbers. Using the theory of generalized Pell's equation, one can easily verify that there is no other class of 2-circular balancing numbers. Hence, the set

$$\{2B_n - 5B_{n-1}, 5B_n - 2B_{n-1} : n = 1,2, \dots\}$$

is an exhaustive list of 2-circular balancing numbers. Each of the two classes of 2-circular balancing numbers can be recurrently calculated by a binary recurrence identical to that for balancing numbers. In particular, these recurrences are

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1}$$

with initial values $x_0 = 5, x_1 = 2, x'_0 = 2, x'_1 = 5$.

We can summarize the above discussion in the following theorem.

6.2.1 Theorem. *The* 2*-circular balancing numbers are solution in* x *of the generalized Pell's equation* $y^2 - 8x^2 = -31$. *These solutions partition in two classes given by* $x_n = 2B_n - 5B_{n-1}$ *and* $x'_n = 5B_n - 2B_{n-1} : n = 1,2, \dots$ *and satisfy the binary recurrences* $x_{n+1} = 6x_n - x_{n-1}$ *and* $x'_{n+1} = 6x'_n - x'_{n-1}$ *with initial values* $x_0 = 5, x_1 = 2, x'_0 = 2$ *and* $x'_1 = 5$.

6.3 3-Circular balancing numbers

In view of the Definition 6.1.1, a natural number x is a 3-circular balancing number if

$$4 + 5 + (x - 1) = (x + 1) + \cdots + m + 1 + 2$$

holds for some natural number m. After simplification, we can conclude that a natural number $x > 3$ is a 3-circular balancing number if and only if $8x^2 - 71$ is a perfect square. Writing $8x^2 - 71 = y^2$, the calculation of 3-circular balancing numbers requires solving of the generalized Pell's equation

$$y^2 - 8x^2 = -71. \tag{6.3}$$

In the last section we have already noticed that the fundamental solution of the Pell's equation $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$ and a fundamental solution of (6.3) is $1 + 3\sqrt{8}$. Using the theory of generalized Pell's equation discussed in Chapter 2, one class of 3-circular balancing numbers is contained in

$$y_n + \sqrt{8}x_n = \left(1 + 3\sqrt{8}\right)\left(3 + \sqrt{8}\right)^{n-1}, \;\; n = 1,2, \dots .$$

Thus, the n^{th} member of this class is given by

$$x_n = \frac{\left(1 + 6\sqrt{2}\right)\left(3 + 2\sqrt{2}\right)^{n-1} - \left(1 - 6\sqrt{2}\right)\left(3 - 2\sqrt{2}\right)^{n-1}}{4\sqrt{2}}, \;\; n = 1,2, \dots .$$

using the Binet form for balancing numbers, it is easy to see that

$$x_n = 3B_n - 8B_{n-1}, n = 1,2, \cdots.$$

It is well-known that $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of the generalized Pell's equation (6.3). Since $x_n = 3B_n - 8B_{n-1}$ and $B_{-n} = -B_n$, it follows that

$$\begin{aligned} x'_n = x_{-n} &= 3B_{-n} - 8B_{-n-1} \\ &= 8B_{n+1} - 3B_n \end{aligned}$$

is positive and greater than 3 for $n = 0,1,2, \dots$ and hence represents another class of 3-circular balancing numbers. One can verify that there are just two fundamental solutions of (6.3). Hence, the set

$$\{3B_n - 8B_{n-1}, 8B_n - 3B_{n-1}: n = 1,2, \dots\}$$

contains all the 3-circular balancing numbers. The two classes of 3-circular balancing numbers can also be expressed by means of the binary recurrences

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1}$$

with initial values $x_0 = 8, x_1 = 3, x'_0 = 3, x'_1 = 8$.

The above discussion proves

6.3.1 Theorem. *The solutions in x of the generalized Pell's equation $y^2 - 8x^2 = -71$ partition in two classes given by $x_n = 3B_n - 8B_{n-1}$ and $x'_n = 8B_n - 3B_{n-1}: n = 1,2,...$ that represent all the 3-circular balancing numbers. These two classes of solutions satisfy the binary recurrences $x_{n+1} = 6x_n - x_{n-1}$ and $x'_{n+1} = 6x'_n - x'_{n-1}$ with initial values $x_0 = 8, x_1 = 3, x'_0 = 3, x'_1 = 8$.*

6.4 4-Circular balancing numbers

By virtue of Definition 6.1.1, a natural number x is a 4-circular balancing number if

$$5 + 6 + (x - 1) = (x + 1) + \cdots + m + 1 + 2 + 3$$

holds for some natural number m. After simplification, it follows that a natural number $x > 4$ is a 4-circular balancing number if and only if $8x^2 - 127$ is a perfect square. Setting $8x^2 - 127 = y^2$, the calculation of 4-circular balancing numbers requires solving the generalized Pell's equation

$$y^2 - 8x^2 = -127. \tag{6.4}$$

We already know that the fundamental solution of the Pell's equation $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$ and a fundamental solution of (6.4) is $1 + 4\sqrt{8}$. Thus, one class of 4-circular balancing numbers is contained in

$$y_n + \sqrt{8}x_n = \left(1 + 4\sqrt{8}\right)\left(3 + \sqrt{8}\right)^{n-1}, n = 1,2, \ldots .$$

Using this equation, the n^{th} member of this class of 4-circular balancing numbers can be written as

$$x_n = \frac{(1+8\sqrt{2})(3+2\sqrt{2})^{n-1} - (1-8\sqrt{2})(3-2\sqrt{2})^{n-1}}{4\sqrt{2}}, n = 1,2, \ldots$$

and referring to the Binet form for balancing numbers, one can get

$$x_n = 4B_n - 11B_{n-1}, n = 1,2, \ldots .$$

As usual, $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of the generalized Pell's equation (6.4). Since $x_n = 4B_n - 11B_{n-1}$ and $B_{-n} = -B_n$, it follows that

$$\begin{aligned} x_n' = x_{-n} &= 4B_{-n} - 11B_{-n-1} \\ &= 11B_{n+1} - 4B_n \end{aligned}$$

is positive and greater than 4 for $n = 0,1,2, \ldots$ and hence represents another class of 4-circular balancing numbers. One can verify that (6.4) has just two fundamental solutions. Therefore, the set

$$\{4B_n - 11B_{n-1}, 11B_n - 4B_{n-1} : n = 1,2, \ldots\}$$

gives the complete list of 4-circular balancing numbers. The two classes of 4-circular balancing numbers can also be recurrently expressed as

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x_{n+1}' = 6x_n' - x_{n-1}'$$

$n = 1,2, \ldots$ with initial values $x_0 = 11, x_1 = 4, x_0' = 4, x_1' = 11$.

In view of the above discussion, we have the following theorem.

6.4.1 Theorem. *The* 4-*circular balancing numbers are solutions in* x *of the generalized Pell's equation* $y^2 - 8x^2 = -127$ *and can be realized in two classes as* $x_n = 4B_n - 11B_{n-1}$ *and* $x_n' = 11B_n - 4B_{n-1} : n = 1,2, \ldots$. *Further, the two classes of* 4-*circular balancing numbers obey the recurrence relations* $x_{n+1} = 6x_n - x_{n-1}$ *and* $x_{n+1}' = 6x_n' - x_{n-1}'$ *with initial values* $x_0 = 11, x_1 = 4, x_0' = 4, x_1' = 11$.

6.5 k-Circular balancing numbers

By virtue of Definition 6.1.1, a natural number $x > k$ is a k-circular balancing number if and only if $8x^2 - 8k^2 + 1$ is a perfect square. Writing $8x^2 - 8k^2 + 1 = y^2$, the k-circular balancing numbers are values of x satisfying the generalized Pell's equation

$$y^2 - 8x^2 = -8k^2 + 1. \tag{6.5}$$

A fundamental solution of the above equation is $1 + k\sqrt{8}$. Thus, one class of k-circular balancing numbers can obtained from

$$y_n + \sqrt{8}x_n = \left(1 + k\sqrt{8}\right)\left(3 + \sqrt{8}\right)^{n-1},\ \ n = 1,2,\dots.$$

The n^{th} member of this class is given by

$$x_n = \frac{\left(1 + 2k\sqrt{2}\right)\left(3 + 2\sqrt{2}\right)^{n-1} - \left(1 - 2k\sqrt{2}\right)\left(3 - 2\sqrt{2}\right)^{n-1}}{4\sqrt{2}}$$

and using the Binet form for balancing numbers, it is easy to see that

$$x_n = kB_n - (3k-1)B_{n-1}, n = 1,2,\dots.$$

It is well-known that $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of (6.5). Since $x_n = kB_n - (3k-1)B_{n-1}$ and $B_{-n} = -B_n$, it follows that

$$\begin{aligned} x'_n = x_{-n} &= kB_{-n} - (3k-1)B_{-n-1} \\ &= (3k-1)B_{n+1} - kB_n \end{aligned}$$

is positive and greater than k for $n = 0, 1, 2, \dots$ and hence gives another class of k-circular balancing numbers. Thus, the set

$$\{kB_n - (3k-1)B_{n-1}, (3k-1)B_n - kB_{n-1} \colon n = 1,2,\dots\}$$

represents two classes of k-circular balancing numbers. These two classes can be recurrently defined as

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1}$$

with initial values $x_0 = 3k - 1,\ x_1 = k,\ x'_0 = k,\ x'_1 = 3k - 1$.

The above discussion can be summarized as follows.

6.5.1 Theorem. *For any arbitrary positive integer* k, *the* k*-circular balancing numbers are solutions in* x *of the generalized Pell's equation* $y^2 - 8x^2 = -8k^2 + 1$. *It is always possible to extract two classes of* k*-circular balancing numbers given by* $x_n = kB_n - (3k-1)B_{n-1}$, $x'_n = (3k-1)B_n - kB_{n-1}: n = 1,2,\dots$. *These two classes can be described in terms of binary recurrences as* $x_{n+1} = 6x_n - x_{n-1}$ *and* $x'_{n+1} = 6x'_n - x'_{n-1}$ *with initial terms* $x_0 = 3k-1$, $x_1 = k$, $x'_0 = k$, $x'_1 = 3k-1$.

6.6 Open problems

It is important to note that two classes of k-circular balancing numbers given in Theorem 6.5.1 may not provide an exhaustive list for some values of k. In particular, the 6-circular balancing numbers are solutions of $y^2 - 8x^2 = -287$ which has four classes of solutions and hence there are four classes of 6-circular balancing numbers. One can verify that these four classes constitutes the set

$$\{6B_n - 17B_{n-1},\ 17B_n - 6B_{n-1},\ 8B_n - 9B_{n-1},\ 9B_n - 8B_{n-1}: n = 1,2,\dots\}.$$

It is not possible to explore all classes of circular balancing numbers for an arbitrary positive integer k as it requires solving the generalized parametrized Pell's equation (6.5). However, there is ample scope for exploring the all k-circular balancing numbers at least for certain sub-classes of natural numbers. We leave it as an open problem for future researchers.

Chapter 7

An Application of Balancing-Like Sequences to a Statistical Diophantine Problem

7.1 Introduction

After the introduction of balancing numbers in 1999 by Behera and Panda [3], many generalizations came up. In 2005, Panda and Ray [32], with a small variation in the definition of balancing numbers, introduced cobalancing numbers. In the year 2012, using a modification in the recurrence relation for balancing numbers, Panda and Rout [37] studied a class of recurrent sequences known as balancing-like sequences. In this thesis, we have also studied certain interesting generalizations of the sequence of balancing numbers namely, the sequences of almost balancing numbers, almost cobalancing numbers, almost balancing-like numbers and circular balancing numbers. After going through so many generalizations of the balancing sequence, one may ask a question, "Is there any relation of any of these sequences with other areas of mathematical science? In this chapter, we answer this question in affirmative by studying a statistical Diophantine problem, the solution of which depends heavily on balancing-like sequences.

7.2 Preliminaries and statement of the problem

The variance of n real numbers $x_1, x_2, \dots, x_n$ is given by $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the mean of $x_1, x_2, \dots, x_n$. Using the above formula, it can be checked that the variance of first n natural numbers (and hence the variance of any n consecutive natural numbers) is ${s_n}^2 = (n^2-1)/12$. It is easy to see that ${s_n}^2$ is a natural number if and only $n \equiv \pm 1 (\bmod 6)$. Our focus is on those values of n that correspond to integral values of the standard deviation s_n. Observe that for some N, s_N is a natural number say, $s_N = \sigma$ if $N^2 - 1 = 12\sigma^2$ which is equivalent to the Pell's equation $N^2 - 12\sigma^2 = 1$. The fundamental solution corresponds to $N_1 = 7$ and $\sigma_1 = 2$. Using the theory of Pell's equation discussed in Chapter 2, we get

$$N_k + 2\sqrt{3}\sigma_k = \left(7 + 4\sqrt{3}\right)^k; k = 1,2, \dots .$$

This gives

$$N_k = \frac{\left(7 + 4\sqrt{3}\right)^k + \left(7 - 4\sqrt{3}\right)^k}{2}$$

and

$$\sigma_k = \frac{\left(7 + 4\sqrt{3}\right)^k - \left(7 - 4\sqrt{3}\right)^k}{4\sqrt{3}},$$

$k = 1,2, \dots$. Since (N_k, σ_k) is a solution of the Pell's equation $N^2 - 12\sigma^2 = 1$, both N_k and σ_k are natural numbers for each k.

7.3 Recurrence relations for N_k and σ_k

In the last section, we obtained the Binet forms for N_k and σ_k where σ_k is the standard deviation of N_k consecutive natural numbers. Notice that the standard deviation of a single number is zero and hence we may assume that $N_0 = 1$ and $\sigma_0 = 0$, and indeed, from the last section, we already have $N_1 = 7$ and $\sigma_1 = 2$. Observe that $u_n = \left(7 + 4\sqrt{3}\right)^n$ and $v_n = \left(7 - 4\sqrt{3}\right)^n$ both satisfy the binary recurrences

$$u_{n+1} = 14u_n - u_{n-1},\ v_{n+1} = 14v_n - v_{n-1};$$

hence, the recurrence relations for the sequences $\{N_k\}$ and $\{\sigma_k\}$ are given by

$$N_{k+1} = 14N_k - N_{k-1};\ N_0 = 1, N_1 = 7$$

and

$$\sigma_{k+1} = 14\sigma_k - \sigma_{k-1};\ \sigma_0 = 0, \sigma_1 = 2.$$

The first few terms of both sequences are thus $N_1 = 7, N_2 = 97, N_3 = 1351, N_4 = 18817$, $N_5 = 262087$ (sequence A011943 in [46]) and $\sigma_1 = 2, \sigma_2 = 28$, $\sigma_3 = 390, \sigma_4 = 5432, \sigma_5 = 75658$ (sequence A011944 in [46]). Using the above binary recurrences for the sequences $\{N_k\}$ and $\{\sigma_k\}$, some useful results can be obtained. The following theorem deals with two identities in which N_k and σ_k behave like hyperbolic functions.

7.3.1 Theorem. *For natural numbers k and l, $\sigma_{k+l} = \sigma_k N_l + N_k \sigma_l$ and $N_{k+l} = N_k N_l + 12\sigma_k \sigma_l$.*

Proof. Since identity

$$N_k + 2\sqrt{3}\sigma_k = \left(7 + 4\sqrt{3}\right)^k$$

holds for each natural number k, it follows that

$$\begin{aligned} N_{k+l} + 2\sqrt{3}\sigma_{k+l} &= \left(7 + 4\sqrt{3}\right)^{k+l} \\ &= \left(7 + 4\sqrt{3}\right)^k \left(7 + 4\sqrt{3}\right)^l \\ &= \left(N_k + 2\sqrt{3}\sigma_k\right)\left(N_l + 2\sqrt{3}\sigma_l\right) \\ &= (N_k N_l + 12\sigma_k\sigma_l) + 2\sqrt{3}(\sigma_k N_l + N_k \sigma_l). \end{aligned}$$

Comparing the rational and irrational parts, the desired results follow. ■

The following corollary is a direct consequence of Theorem 7.3.1.

7.3.2 Corollary. *For any natural number k, $\sigma_{k+1} = 7\sigma_k + 2N_k, N_{k+1} = 7N_k + 24\sigma_k$, $\sigma_{2k} = 2\sigma_k N_k$ and $N_{2k} = {N_k}^2 + 12{\sigma_k}^2$.*

Theorem 7.3.1 can be used for the derivation of another similar result. The following theorem provides formulas for σ_{k-l} and N_{k-l} in terms of N_k, N_l, σ_k and σ_l.

7.3.3 Theorem. *If k and l are natural numbers with $k > l$, then $\sigma_{k-l} = \sigma_k N_l - N_k \sigma_l$ and $N_{k-l} = N_k N_l - 12\sigma_k \sigma_l$*

Proof. By virtue of Theorem 7.3.1,

$$\sigma_k = \sigma_{(k-l)+l} = \sigma_{k-l} N_l + N_{k-l} \sigma_l$$

and

$$N_k = N_{(k-l)+l} = 12\sigma_{k-l}\sigma_l + N_{k-l}N_l.$$

Solving these two equations for σ_{k-l} and N_{k-l}, we obtain

$$\sigma_{k-l} = \frac{\begin{vmatrix} \sigma_k & \sigma_l \\ N_k & N_l \end{vmatrix}}{\begin{vmatrix} N_l & \sigma_l \\ 12\sigma_l & N_l \end{vmatrix}} = \frac{\sigma_k N_l - N_k \sigma_l}{{N_l}^2 - 12{\sigma_l}^2}$$

and

$$N_{k-l} = \frac{\begin{vmatrix} N_l & \sigma_k \\ 12\sigma_l & N_k \end{vmatrix}}{\begin{vmatrix} N_l & \sigma_l \\ 12\sigma_l & N_l \end{vmatrix}} = \frac{N_k N_l - 12\sigma_k \sigma_l}{{N_l}^2 - 12{\sigma_l}^2}.$$

Since for each natural number l, (N_l, σ_l) is a solution of the Pell's equation $N^2 - 12\sigma^2 = 1$, the proof is complete. ■

The following corollary follows from Theorem 7.3.3 in the exactly same way Corollary 7.3.2 follows from Theorem 7.3.1.

7.3.4 Corollary. *For any natural number* $k > 1$, $\sigma_{k-1} = 7\sigma_k - 2N_k$ *and* $N_{k-1} = 7N_k - 24\sigma_k$.

Theorems 7.3.1 and 7.3.3 can be utilized to form interesting higher order non-linear recurrences for the sequences $\{N_k\}$ and $\{\sigma_k\}$. The following theorem is crucial in this regard.

7.3.5 Theorem. *If k and l are natural numbers with* $k > l$, $\sigma_{k-l} \cdot \sigma_{k+l} = {\sigma_k}^2 - {\sigma_l}^2$ *and* $N_{k-l} \cdot N_{k+l} + 1 = {N_k}^2 + {N_l}^2$.

Proof. By virtue of Theorems 7.3.1 and 7.3.3,

$$\sigma_{k-l} \cdot \sigma_{k+l} = {\sigma_k}^2 {N_l}^2 - {N_k}^2 {\sigma_l}^2$$

and since for each natural number r, ${N_r}^2 = 12{\sigma_r}^2 + 1$,

$$\begin{aligned}\sigma_{k-l} \cdot \sigma_{k+l} &= {\sigma_k}^2(12{\sigma_l}^2 + 1) - {\sigma_l}^2(12{\sigma_k}^2 + 1) \\ &= {\sigma_k}^2 - {\sigma_l}^2.\end{aligned}$$

Further,

$$\begin{aligned}N_{k-l} \cdot N_{k+l} &= {N_k}^2{N_l}^2 - 144{\sigma_k}^2{\sigma_l}^2 \\ &= {N_k}^2{N_l}^2 - 144 \cdot \frac{{N_k}^2 - 1}{12} \cdot \frac{{N_l}^2 - 1}{12}\end{aligned}$$

implies

$$N_{k-l} \cdot N_{k+l} + 1 = {N_k}^2 + {N_l}^2. \quad \blacksquare$$

The following corollary is a direct consequence of Theorem 7.3.5.

7.3.6 Corollary. *For any natural number* $k > 1$, $\sigma_{k-1} \cdot \sigma_{k+1} = {\sigma_k}^2 - 4$ *and* $N_{k-1} \cdot N_{k+1} = {N_k}^2 + 48$.

In view of Theorem 7.3.5, we also have ${\sigma_{k+1}}^2 - {\sigma_k}^2 = 2\sigma_{2k+1}$. Adding this identity for $k = 0,1, \dots, l-1$, we get

$$2(\sigma_1 + \sigma_3 + \cdots + \sigma_{2l-1}) = {\sigma_l}^2.$$

The above discussion proves

7.3.7 Corollary. *Twice the sum of first l odd indexed terms of the standard deviation sequence is equal to the variance of first N_l natural numbers.*

The following corollary is also a direct consequence of Theorem 7.3.5.

7.3.8 Corollary. *For each natural number* k, $7(N_1 + N_3 + \cdots + N_{2k-1}) + k = 2(N_1^2 + N_2^2 + \cdots + N_{k-1}^2) + N_k^2$.

7.4 Balancing-like sequences derived from $\{N_k\}$ and $\{\sigma_k\}$

The linear binary recurrences for the sequences $\{N_k\}$ and $\{\sigma_k\}$ along with their properties suggest that $\{\sigma_k/2\}$ is a balancing-like sequence whereas $\{N_k\}$ is the corresponding Lucas-balancing-like sequence [37]. In addition, these sequences are closely related to two other sequences that can also be described by similar binary recurrences.

The following theorem deals with a sequence derived from $\{N_k\}$, the terms of which are factors of corresponding terms of the sequence $\{\sigma_k\}$.

7.4.1 Theorem. *For each natural number k, $(N_k+1)/2$ is a perfect square. Further, $M_k=\sqrt{(N_k+1)/2}$ divides σ_k.*

Proof. By virtue of Theorem 7.3.1 and the Pell's equation $N^2-12\sigma^2=1$

$$\frac{N_{2k}+1}{2}=\frac{{N_k}^2+12{\sigma_k}^2+1}{2}={N_k}^2$$

implying that $M_{2k}=N_k$. Since $\sigma_{2k}=2\sigma_k N_k$, M_{2k} divides σ_{2k} for each natural number k. Further,

$$\begin{aligned}\frac{N_{2k+1}+1}{2}&=\frac{7N_{2k}+24\sigma_{2k}+1}{2}\\&=\frac{7\left({N_k}^2+12{\sigma_k}^2\right)+48\sigma_k N_k+1}{2}\\&=84{\sigma_k}^2+24\sigma_k N_k+4\\&=36{\sigma_k}^2+24\sigma_k N_k+4{N_k}^2\\&=(6\sigma_k+2N_k)^2\\&=(7\sigma_k+2N_k-\sigma_k)^2\\&=(\sigma_{k+1}-\sigma_k)^2\end{aligned}$$

from which we obtain $M_{2k+1}=\sigma_{k+1}-\sigma_k$. By virtue of Theorem 7.3.5, ${\sigma_{k+1}}^2-{\sigma_k}^2=2\sigma_{2k+1}$ and thus

$$\begin{aligned}\sigma_{2k+1}&=\frac{\sigma_{k+1}+\sigma_k}{2}\cdot(\sigma_{k+1}-\sigma_k)\\&=\delta_k(\sigma_{k+1}-\sigma_k)\end{aligned}$$

where $\delta_k = \frac{\sigma_{k+1}+\sigma_k}{2}$ is a natural number since σ_k is even for each k and hence M_{2k+1} divides σ_{2k+1}. ■

We have shown, while proving Theorem 7.4.1, that $M_{2k+1} = \sigma_{k+1} - \sigma_k$. This proves

7.4.2 Corollary. *The sum of first l odd terms of the sequence $\{M_k\}$ is equal to the standard deviation of the first N_l natural numbers.*

By virtue of Theorem 7.4.1, M_k divides σ_k for each natural number k. Therefore, it is natural to study the sequence $L_k = \sigma_k / M_k\,, k = 1,2, \ldots$. From the proof of Theorem 7.4.1, it follows that $L_{2k} = 2\sigma_k$ and $L_{2k+1} = (\sigma_{k+1} + \sigma_k)/2$.

Our next objective is to show that the sequence $\{L_k\}_{k=1}^{\infty}$ is a balancing-like sequence and $\{M_k\}_{k=1}^{\infty}$ is the corresponding Lucas-balancing-like sequence. This claim is validated by the following theorem.

7.4.3 Theorem. *For each natural number k, ${M_k}^2 = 3{L_k}^2 + 1$. Further, the sequences $\{L_k\}_{k=1}^{\infty}$ and $\{M_k\}_{k=1}^{\infty}$ satisfy the binary recurrences $L_{k+1} = 4L_k - L_{k-1}, k \geq 1$ with $L_0 = 0$ and $L_1 = 1$ and $M_{k+1} = 4M_k - M_{k-1}, k \geq 1$ with $M_0 = 1$ and $M_1 = 2$.*

Proof. In view of the Pell's equation $N^2 - 12\sigma^2 = 1$, Corollary 7.4.4 and the discussion following Corollary 7.4.2,

$$\begin{aligned} 3{L_{2k}}^2 + 1 &= 3(2\sigma_k)^2 + 1 \\ &= {N_k}^2 = {M_{2k}}^2 \end{aligned}$$

and

$$\begin{aligned} 3{L_{2k-1}}^2 + 1 &= 3\left(\frac{\sigma_k + \sigma_{k-1}}{2}\right)^2 + 1 \\ &= 3(4\sigma_k - N_k)^2 + 1 \\ &= (6\sigma_k - 2N_k)^2 \\ &= (\sigma_k - \sigma_{k-1})^2 = {M_{2k-1}}^2. \end{aligned}$$

To this end, using Corollary 7.3.2, we get

$$\begin{aligned} 4M_{2k+1} - M_{2k} &= 4(\sigma_{k+1} - \sigma_k) - N_k \\ &= 4(6\sigma_k + 2N_k) - N_k \\ &= N_{k+1} = M_{2k+2} \end{aligned}$$

and

$$\begin{aligned} 4M_{2k} - M_{2k-1} &= 4N_k - (\sigma_{k+1} - \sigma_k) \\ &= 4N_k - (-6\sigma_k + 2N_k) \\ &= 6\sigma_k + 2N_k \\ &= \sigma_{k+1} - \sigma_k = M_{2k+1}. \end{aligned}$$

Thus, the sequence M_k satisfies the binary recurrence $M_{k+1} = 4M_k - M_{k-1}$. Similarly, the identities

$$\begin{aligned} 4L_{2k+1} - L_{2k} &= 2(\sigma_{k+1} + \sigma_k) - 2\sigma_k \\ &= 2\sigma_{k+1} = L_{2k+2} \end{aligned}$$

and

$$\begin{aligned} 4L_{2k} - L_{2k-1} &= 8\sigma_k - \frac{\sigma_k + \sigma_{k-1}}{2} \\ &= 8\sigma_k - (4\sigma_k - N_k) \\ &= 4\sigma_k + N_k \\ &= \frac{\sigma_{k+1} + \sigma_k}{2} = L_{2k+1} \end{aligned}$$

confirm that the sequence L_k satisfies the binary recurrence $L_{k+1} = 4L_k - L_{k-1}$. ■

It is easy to check that the Binet forms of the sequences $\{L_k\}$ and $\{M_k\}$ are respectively

$$L_k = \frac{\left(2+\sqrt{3}\right)^k - \left(2-\sqrt{3}\right)^k}{2\sqrt{3}}$$

and

$$M_k = \frac{\left(2+\sqrt{3}\right)^k + \left(2-\sqrt{3}\right)^k}{2}$$

$k = 1,2,\cdots$. Using the Binet forms or otherwise, the interested reader is invited to verify the following identities.

(a) $L_1 + L_3 + \cdots + L_{2n-1} = {L_n}^2$,

(b) $M_1 + M_3 + \cdots + M_{2n-1} = L_{2n}/2$,

(c) $L_2 + L_4 + \cdots + L_{2k} = L_k L_{k+1}$,

(d) $M_2 + M_4 + \cdots + M_{2k} = (L_{2k+1} - 1)/2$,

(e) $L_{x+y} = L_x M_y + M_x L_y$,

(f) $M_{x+y} = M_x M_y + 3L_x L_y$.

7.5 Scope for further research

In Section 3, we have noticed that $x_k = \frac{\sigma_k}{2}, k = 0, 1, 2, \ldots$ is a balancing-like sequence with binary recurrence $x_{k+1} = 14x_k - x_{k-1}, x_0 = 0, x_1 = 1$. The corresponding Lucas-balancing-like sequence is $y_k = N_k, k = 0, 1, 2, \ldots$ which satisfies a similar binary recurrence $y_{k+1} = 14y_k - y_{k-1}$, of course with different initial values $y_0 = 1$ and $y_1 = 7$. The beauty lies in these pair of balancing-like and Lucas-balancing-like sequences is that $2x_k$ is the standard deviation of $1, 2, \ldots, y_k$ (or any y_k consecutive integers), $k = 1, 2, \ldots$. It is an open problem to investigate the association of such sequences in other statistical problems, for example, one may be interested in integral mean deviation, or covariance of some bivariate distribution.

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List of Publications

1. G. K. Panda and A. K. Panda, *Balancing-like sequences associated with integral standard deviations of consecutive natural numbers*, Fibonacci Quart., 52(5) (2014), 187-192.

2. G. K. Panda and A. K. Panda, *Almost balancing numbers*, J. Indian Math. Soc., 82(3-4) (2015), 147-156.